TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 62 NUMBER 2

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FEBRUARY 1955

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

SOME MATHEMATICAL ASPECTS OF SWITCHING

FRANZ HOHN, Bell Telephone Laboratories and University of Illinois

1. Introduction. In electronic digital computers, telephone switching systems, control systems for automatic factories, and other systems involving the communication or processing of data, one finds many examples of electric circuits which employ what are known as *two-state* or *bi-stable devices*. The simplest example of such a device is a switch or *contact*. When a contact is operated with the aid of an electromagnet, the combination is a called a *relay*. A switch or relay is called a *bilateral circuit element* since it permits the passage of current in either direction when it is closed.

Some commonly used symbols for contacts are shown in Figure 1. In this paper we shall often use the first type of symbol, but usually without drawing in the electromagnet which operates the springs of the contact. (Note that the contacts illustrated here have two *springs*—the one attracted by the electromagnet presses against the other.) A normally open contact is called a "make" or "front" contact whereas a normally closed one is called a "break" or "back" contact.

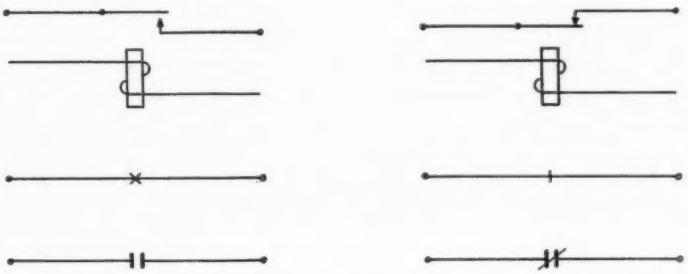


FIG. 1

There are various other two-state devices in use and in the process of development. These include rectifying diodes, magnetic cores, transistors, various types of electron tubes, and others. Magnetic drums and magnetic tape may be considered to be assemblages of two-state devices. The nature of the two "states" varies from one device to another, and includes conducting *vs.* non-conducting, closed *vs.* open, charged *vs.* discharged, magnetized *vs.* non-magnetized, high-potential *vs.* low-potential, and other conditions.

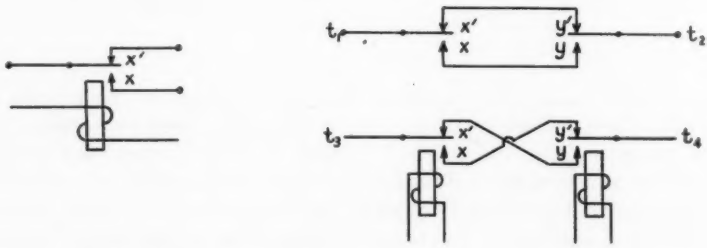
The methods and results of Boolean algebra and related subjects have been found useful in discussing circuits employing two-state devices. In this paper we use contact networks to illustrate how this is done. Naturally, there are many aspects of the design, development, and use of two-state devices which are the special responsibility of the electrical engineer. However, there are other aspects of their use which are essentially abstract in nature, and are therefore

legitimately of interest to the mathematician. The latter provide a neat and novel example of applied mathematics which it is the purpose of this paper to present to the reader in an informal way. We shall also indicate some problems which are worthy of attention and give selected references for a further study of the subject. As far as theoretical results are concerned, the paper presents nothing new—most of what is presented here may also be found in Shannon [3], [4], and Keister, Ritchie, Washburn [2].

2. A mathematical model of a two-terminal relay switching circuit. We introduce two symbols as the first step in constructing the desired model. With an open contact or path in a circuit we associate the symbol "0" and with a closed contact or path we associate the symbol "1." Although we call these symbols "zero" and "one" respectively, they are not the zero and one of ordinary arithmetic, as will presently be clear.

When the condition of a contact is variable in a problem, we call it a *circuit variable* and denote it by a literal symbol such as a, b, x, y, $\cdots$. This symbol takes on the value 0 when the contact is open, the value 1 when it is closed. With each symbol x, we associate a symbol x' to denote a contact which is open when x is closed and closed when x is open.

When two or more contacts always open and close simultaneously, we denote them by the same symbol throughout the circuit. For example, we ordinarily assume that two "make" (or two "break") contacts operated by the same electromagnet open and close simultaneously and hence we denote them by the same symbol. If a make contact is denoted by x, then a break contact operated by the same electromagnet is denoted by x'. Frequently, in order to save one spring, a make and a break are combined in one device called a *transfer*. The points just mentioned are illustrated in Figure 2A.



Transfer Contact

A B

FIG. 2

Employing the notation above defined, the reader may verify that in Figure 2B, there is a path from t_1 to t_2 if and only if $x'=y'=1$ or $x=y=1$, and a path from t_3 to t_4 if and only if $x=0, y=1$, or $x=1, y=0$.

Figure 2A reveals one respect in which our mathematical model is a fiction. When the coil of such a transfer contact is energized, there is a brief instant when neither contact is closed. Thus the assumption that we may call one contact x and the other x' is not strictly justified. For many purposes, this matter is not serious. When it is, one must use design techniques which will take care of any resulting difficulties. These techniques include the use of fast or slow acting relays, make-before-break contacts, *etc.*

Another assumption which we make is that when the coil of a relay is energized, its contacts are instantaneously operated. This assumption too at times causes difficulties which may be taken care of by treating the coil as one two-state device (energized or not energized) and a contact on it as another (operated or not operated). In this paper, we shall restrict our attention to circuits such that these difficulties may be ignored.

Next we introduce two operations on the symbols 0, 1, x, $\cdots$. The operation of "addition," denoted by "+," corresponds to connecting contacts or combinations thereof in parallel. (In the figure of Postulate 1a, contacts a and b are shown connected in parallel.) The operation of "multiplication," denoted by "$\cdot$" or juxtaposition, corresponds to connecting contacts or combinations thereof in series. (In the figure of Postulate 1b, contacts a and b are shown connected in series.) The postulates governing the use of the operations $+$, $\cdot$, $'$, are summarized in Table 1, in which two circuits are indicated as *equivalent* ($\sim$) if and only if they are open and closed under the same conditions of the circuit variables. The circuit diagrams are intended to indicate how the postulates are suggested by physical considerations. The notation is similar to that used by Shannon [3], but the meanings of $+$ and $\cdot$ and also of 0 and 1 are interchanged. Our notation conforms to the widest current practice and is the most convenient when it comes to adapting these ideas to multi-terminal and electronic circuits [15], [10].

From this table of postulates, it is clear that the operation of addition has the meaning "or." In fact, the circuits of (1a) provide a path if a is closed or b is closed (or both are closed). On the other hand, the operation of multiplication has the meaning "and." Thus the circuits of (1b) provide a path if and only if both *a and b* are closed.

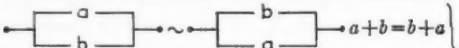
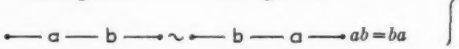
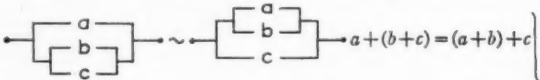
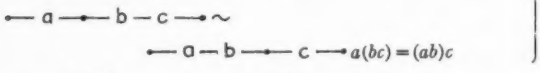
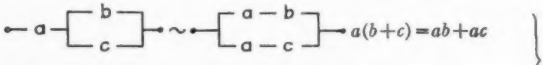
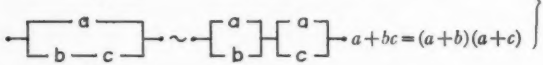
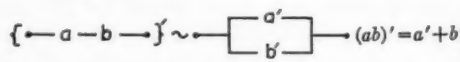
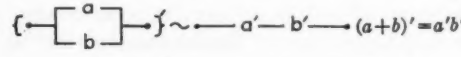
Another important fact to note is that these laws, except for (9), appear in *dual pairs*, each member of a pair being obtainable from the other by a simple interchange of the operations $+$ and $\cdot$. In these simple cases, this interchange corresponds to the interchange of series and parallel connections.

Next we observe that there is a large class of two-terminal switching circuits with which we can associate a function f whose characteristic property is that it takes on the value 1 for those combinations of values of the circuit variables which correspond to the circuit's being closed, but is 0 otherwise. This function is called the *switching function* of the circuit.

In certain simple cases the switching function has in effect already been defined: this is shown in Table 2. Now consider any two-terminal switching

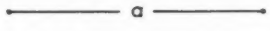
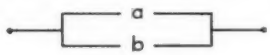
TABLE 1

THE POSTULATES OF SWITCHING ALGEBRA

	Circuit Diagram	Symbolic Equivalent	Verbal Statement
(1a)		$a+b=b+a$	The commutative laws.
(1b)		$ab=ba$	
(2a)		$a+(b+c)=(a+b)+c$	The associative laws.
(2b)		$a(bc)=(ab)c$	
(3a)		$a(b+c)=ab+ac$	The distributive laws.
(3b)		$a+bc=(a+b)(a+c)$	
(4a)		$a+a=a$	The idempotent laws.
(4b)		$a\cdot a=a$	
(5a)		$0+a=a$	The laws of operation with 0.
(5b)		$0\cdot a=0$	
(6a)		$1+a=1$	The laws of operation with 1.
(6b)		$1\cdot a=a$	
(7a)		$a\cdot a'=0$	The laws of complementarity.
(7b)		$a+a'=1$	
(8a)		$(ab)'=a'+b'$	The dualization laws.
(8b)		$(a+b)'=a'b'$	
(9)		$(a')'=a$	The law of involution.

circuit built up solely of series-parallel connected contacts. (The circuits shown in Table 2 are all series-parallel. So is any circuit obtainable by substituting a known series-parallel circuit for any contact of a known series-parallel circuit,

TABLE 2

Circuit	Switching Function
	0
	1
	a
	$a+b$
	ab

or by a series of such substitutions. The symbols used to name the contacts are of course arbitrary. Non-series-parallel circuits are called "bridges.") With each contact we associate a symbol as before, designating identically behaving contacts by the same symbol and oppositely behaving contacts by a symbol and its

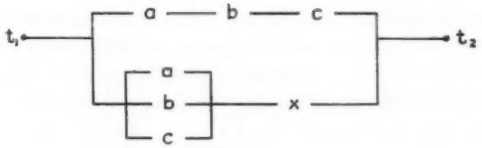


FIG. 3

prime, but using distinct symbols otherwise. Then, using the operations + and ·, we can associate with the circuit a polynomial in the circuit variables which expresses formally the various series and parallel connections. For example, with the circuit of Figure 3, we can associate the switching function

$$= (a + b + c)(abc + x) = abc + (a + b + c)x.$$

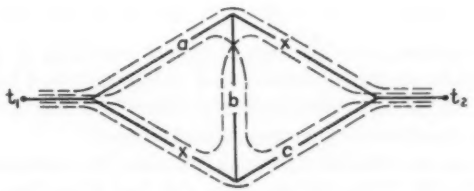


FIG. 4

We can also associate such a function with a bridge-type circuit by first determining, in effect, an equivalent series-parallel circuit. For example, the bridge circuit of Figure 4 provides the possible paths ax, $xbx = bx$, cx, and abc

between the terminals t_1 and t_2. These are just the same paths as are provided by the first-given series-parallel circuit, so that the function f may also be considered as the switching function of this bridge. It should be observed that the above bridge realizes the same switching function as the original series-parallel network, but with two fewer contacts.

With this extension of our notation to bridge circuits, the meanings of "$\cdot$" and "$+$" are no longer restricted to simple series and parallel connections, respectively. Indeed, in the function f, these operations are now to be given the meanings "and" and "or" mentioned earlier. In the example just given, there is a path through the circuit if

a and b and c are 1

or if

a or b or c is 1 *and x is* 1.

Thus the switching function now describes the *electrical behavior* but not necessarily the *geometry* of the network.

This procedure may be applied to any two-terminal series-parallel or bridge network which may be represented diagrammatically as in Figure 5. Here the box is assumed to contain all the contacts of the circuit, which we denote by $x_1, \cdots, x_n$ and their primes. These contacts are assumed to be operated by the corresponding electromagnets $X_1, X_2, \cdots, X_n$, which are controlled from

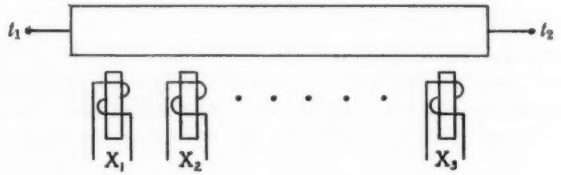


FIG. 5

outside the box. In this case the open or closed condition of the path between t_1 and t_2 depends only on the combination of values assumed by the circuit variables $x_1, \cdots, x_n$, in terms of which the switching function is expressed. Such a circuit is called a *combinational* two-terminal circuit.

It is important to note that the switching functions in n variables $x_1, x_2, \cdots, x_n$, *satisfy the laws* (1)–(9). This follows from the fact that they assume only the values 0 or 1, for which these laws have been postulated to hold.

We have one last definition to make, and then our model is complete. Let f_1 and f_2 be functions associated in the above manner with two switching circuits s_1 and s_2. If s_1 is never closed unless s_2 is also closed, then whenever f_1 is 1, f_2 is 1

also (although when $f_1=0$, we may have $f_2=1$). When f_1 and f_2 are thus related, we write $f_1 \leqq f_2$ or $f_2 \geqq f_1$, reading the symbols in the usual manner. Then for any switching functions, f, g, h we have the following rules:

(10) $0 \leqq f \leqq 1$ The universal bounds property.

(11) $f \leqq f$ The reflexive property.

(12) If $f \leqq g$ and $g \leqq f$, then $f = g$ The anti-symmetric property.

(13) If $f \leqq g$ and $g \leqq h$, then $f \leqq h$ The transitive property.

(14a) $f \leqq g$ if and only if $fg = f$

(14b) $f \leqq g$ if and only if $f + g = g$ } The consistency principle.

The fact that the switching functions in n variables satisfy the rules (1)–(14) demonstrates that these functions as a class constitute what is known as a *Boolean algebra*. Actually, a much shorter list of rules would suffice to establish this fact. However, the list given here is extensive enough to include the most fundamental properties. It is precisely the list given in Birkhoff and MacLane, *A Survey of Modern Algebra*, Chapter XI, except for some changes in notation which conform to the most common engineering usage.

In discussions of Boolean algebra, the functions of n variables and their primes built up with the aid of the operations $+$, $\cdot$, and $'$ are called *Boolean functions*. We shall continue to call them "switching functions" here.

3. Some useful identities. It is possible to establish an unlimited number of formal identities in Boolean algebra, starting with the laws (1) to (9). These may be obtained by algebraic methods or by "complete induction," that is, by checking the correctness of the identity for all possible combinations of values of the variables. Often such identities are suggested by network considerations.

We illustrate with two identities which are extremely useful in simplifying switching functions. The first of these is

(15a) $$x + xy = x.$$

This is proved with the aid of (6b), (3a), and (6a) as follows:

$$x + xy = x\cdot 1 + xy = x(1 + y) = x\cdot 1 = x.$$

Alternatively, we have the readily computed table of values:

x	y	$x + xy$
0	0	0
0	1	0
1	0	1
1	1	1

Since the first and third columns are the same for all combinations of the variables, we conclude that $x+xy=x$. Finally, the circuits in Figure 6 are both closed if and only if x is 1. Hence $x+xy=x$.

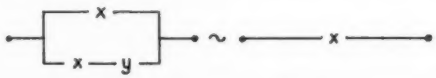


FIG. 6

The dual of (15a) is

$$x(x + y) = x. \tag{15b}$$

Another extremely useful identity of this type is

$$x + x'y = x + y. \tag{16a}$$

Here we have, by (3b) and (7b),

$$x + x'y = (x + x')(x + y) = 1 \cdot (x + y) = x + y.$$

Circuitwise, this also makes sense, as Figure 7 indicates. In fact, if $x=1$, each circuit is closed, regardless of the value of y. If $x=0$, then $x'=1$, so that each

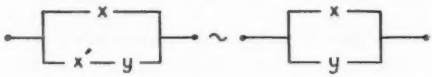


FIG. 7

circuit is then closed only when $y=1$. Thus the circuits are closed under the same conditions and hence are equivalent.

The dual of (16a) is

$$x(x' + y) = xy. \tag{16b}$$

Other useful identities are

$$(x + y)(x' + z)(y + z) = (x + y)(x' + z) \tag{17a}$$

$$xz + x'y + yz = xz + x'y \tag{17b}$$

$$(x + y)(x' + z) = xz + x'y. \tag{18}$$

The reader may enjoy deriving these.

To see how these identities are used, consider the simplification of the series-parallel circuit whose switching function is

$$f = abc + ab'c + a'b'c.$$

We have

$$\begin{aligned} f &= c(ab + ab' + a'b') \\ &= c[a(b + b') + a'b'] \\ &= c(a + a'b') \\ &= c(a + b'). \end{aligned}$$

The circuits are given in Figure 8. The simplification saves 6 contacts. An important thing to notice is that circuitwise, $c(a+b')$ is simpler than $ca+cb'$

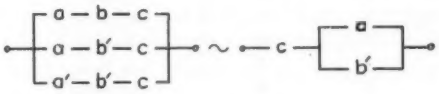


FIG. 8

since one less contact is indicated. In the case of a complicated function, it is not always easy to determine the factorization which is the simplest from the point of view of "hardware." (See, however, [16].)

4. The canonical expansions. Of particular interest among the switching functions of n variables are the products containing all n of the variables as factors, either primed or not (but not both primed and unprimed in any one case, of course). When $n=1$, we consider x_1 and x_1' to be these "products." When $n=2$, they are $x_1'x_2'$, $x_1'x_2$, x_1x_2', and x_1x_2. In general, since each of the variables is chosen in primed or unprimed form, the number of such products is 2^n. We call these the "fundamental products" of the n variables. The important property of a fundamental product is that it takes on the value 1 for exactly one set of values of x_1 to x_n, namely that combination which makes each factor of the product equal to 1.

If we define $x_j^0 = x_j'$ and $x_j^1 = x_j$, then a fundamental product may be represented in the form $x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where each α is either 0 or 1. It is convenient to interpret the sequences of superscripts $\alpha_1\ \alpha_2 \cdots \alpha_n$ as integers in binary notation. Then we use the corresponding decimal integers to number the fundamental products as follows:

$$p_i = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where

$$(i)_{\text{decimal}} = (\alpha_1\alpha_2 \cdots \alpha_n)_{\text{binary}}.$$

We may employ the same scheme to number the *fundamental sums*:

$$s_i = x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n}.$$

The characteristic property of such a sum is that it vanishes for exactly one combination of values of the x's, namely the one which makes each term of the sum equal 0.

The points made thus far are illustrated for $n=3$ in Table 3.

TABLE 3

i	x_1	x_2	x_3	Non-Vanishing Product, p_i	Vanishing Sum, s_{7-i}
0	0	0	0	$x_1' x_2' x_3'$	$x_1 + x_2 + x_3$
1	0	0	1	$x_1' x_2' x_3$	$x_1 + x_2 + x_3'$
2	0	1	0	$x_1' x_2 x_3'$	$x_1 + x_2' + x_3$
3	0	1	1	$x_1' x_2 x_3$	$x_1 + x_2' + x_3'$
4	1	0	0	$x_1 x_2' x_3'$	$x_1' + x_2 + x_3$
5	1	0	1	$x_1 x_2' x_3$	$x_1' + x_2 + x_3'$
6	1	1	0	$x_1 x_2 x_3'$	$x_1' + x_2' + x_3$
7	1	1	1	$x_1 x_2 x_3$	$x_1' + x_2' + x_3'$

It is a simple matter to express a given switching function as a sum of fundamental products. For example, suppose $n=3$ and

$$f = (x_1' x_2)'(x_1 + x_3).$$

First we express f as a sum of products, not necessarily fundamental:

$$f = (x_1 + x_2')(x_1 + x_3) = x_1 + x_2' x_3.$$

Then by (7b) we have

$$f = x_1(x_2' + x_2)(x_3' + x_3) + (x_1 + x_1')x_2' x_3.$$

Now expanding and removing duplicate terms by (4a), we obtain

$$f = x_1 x_2' x_3' + x_1 x_2' x_3 + x_1 x_2 x_3' + x_1 x_2 x_3 + x_1' x_2' x_3,$$

which contains only distinct fundamental products. The method is obviously perfectly general. Once we have expanded a switching function into such a sum of fundamental products, we can read off by inspection of the terms present the combinations of values of the n variables for which the function is 1.

Conversely, suppose we know all the combinations of values of the n x's for which a switching function is 1. Using the fact that a fundamental product takes on the value 1 for exactly one such combination one can easily prove the following theorem ([1], p. 344):

Every switching function f of n variables may be expressed uniquely as a sum of fundamental products of these n variables:

$$f = \sum_{i=0}^{2^n-1} f_i p_i$$

where $f_i = 0$ if $f = 0$ for the combination that makes $p_i = 1$, but $f_i = 1$ if $f = 1$ for this combination.

In other words, f may be written as the sum of the fundamental products corresponding to those combinations of values of the x_j's for which $f=1$. This sum is called the *canonical expansion of f*.

From this result, since each of the 2^n fundamental products has a coefficient 0 or 1 in the canonical expansion, we may conclude as a corollary:

There are 2^{2^n} switching functions of n variables.

The canonical expansion is useful for translating verbally stated conditions into algebraic form. Suppose, for example, we wish to design a two-terminal circuit using contacts on three relays and satisfying the following requirements:

"The network is to be closed whenever the relay x is released unless y or z (but not both) is operated, in which case it is open. It is also to be closed when x is operated unless y and z are both operated or both released, in which case it is open."

These conditions are summarized in column "f", Table 4. (The columns "g" and "h" are for later illustrations.) We have, from the table, with the aid of the above theorem,

$$f = x'y'z' + x'yz + xy'z + xyz'.$$

That is, in this case, $f_0=f_3=f_5=f_6=1$ and $f_1=f_2=f_4=f_7=0$. This is an example of a *symmetric function*. It is unchanged by any permutation of x, y, z. (The

TABLE 4

i	x	y	z	f	g	h
0	0	0	0	1	0	0
1	0	0	1	0	1	1
2	0	1	0	0	0	0
3	0	1	1	1	1	1
4	1	0	0	0	1	d
5	1	0	1	1	1	1
6	1	1	0	1	1	d
7	1	1	1	0	0	d

primes are not permuted along with the variables.) The series-parallel realization of this function requires 12 contacts unless some simplification is effected. If we write

$$f = x'(y'z' + yz) + x(y'z + yz'),$$

we can obtain a series-parallel realization with only 10 contacts, as shown in Figure 9A. However, the use of a bridge circuit allows us to realize the function with only 8 contacts. (Figure 9B).

Unfortunately, there seems to be no way of looking at the switching function itself and seeing that a bridge circuit is most economical of contacts.

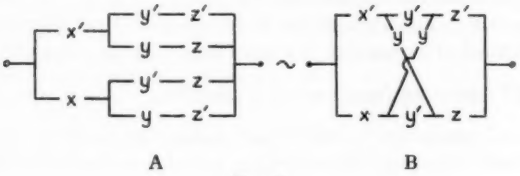


FIG. 9

In the actual construction of this circuit, transfer contacts would be employed in order to save springs. (Figure 10.)

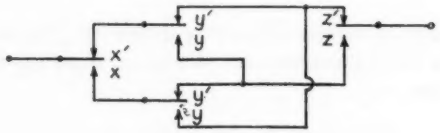


FIG. 10

Symmetric circuits, of which this is an example, are discussed in detail by Shannon [3] and in the book by Keister, Ritchie, and Washburn [2].

It may be shown that in addition to the canonical expansion, a switching function has a *dual canonical expansion*:

Every switching function g of n variables may be expressed uniquely as a product of fundamental sums:

$$g = \prod_{i=0}^{2^n-1} (g_i + s_i),$$

where $g_i = 0$ *if* $g = 0$ *for the combination of values which makes* $s_i = 0$, *but* $g_i = 1$ *if* $g = 1$ *for this combination.*

That is, g may be written as the product of those vanishing sums corresponding to the combinations of values of the x's for which $g = 0$ since when $g_i = 1$, $g_i + s_i = 1$ also.

The dual canonical expansion is sometimes more convenient than the canonical expansion for expressing a function in algebraic form. For example, from Table 4, column "g", and the preceding theorem we have

$$g = (x_1 + x_2 + x_3)(x_1 + x_2' + x_3)(x_1' + x_2' + x_3'),$$

which reduces to

$$g = x_1(x_2' + x_3') + x_1' x_3$$

with the aid of (3b), (7a), and (17b).

Finally, consider Table 4, column "h". The entries "d" in this column denote the fact that we don't care whether 0's or 1's appear in those positions. (In practice this situation arises when it follows from the conditions the circuit is to be designed to satisfy that the combinations of values of the x's labelled "d" do not occur in the use of the circuit in question.) We should therefore choose values for the d's in such a way as to render the function h as simple as possible. In this case, if we make 0's of the first two d's and 1 of the last, the h-column coincides with the x_3-column, so that

$$h = x_3.$$

Thus the circuit requires the use of only one contact.

From the preceding examples, it is clear that one method of synthesizing a two-terminal relay contact network is this:

(a) Determine the requirements the circuit is to satisfy.
(b) Express the requirements algebraically in the form of a switching function.
(c) Simplify the switching function as much as possible, using relations (1)–(18) as they apply, and any "don't-care" conditions which may exist.
(d) Draw the circuit and see what further simplifications can be made.

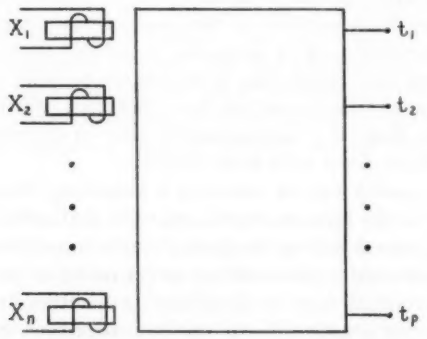


FIG. 11

5. Combinational circuits. We have been dealing up to this point exclusively with two-terminal circuits. Frequently, however, more terminals are required. We might, for example, need to design a circuit which establishes prescribed connections between terminals $t_1, t_2, \cdots, t_p$ for specified combinations of values of the variables $x_1, x_2, \cdots, x_n$, the corresponding contacts being operated respectively by coils $X_1, X_2, \cdots, X_n$, whose control is independent of the circuit in question. A circuit of this kind may be represented as in Figure 11. As before,

the box is presumed to contain all the contacts of the circuit, these being operated solely by the electromagnets $X_1, X_2, \cdots, X_n$, which are controlled from outside the box. Such a circuit is again called a *combinational circuit.*

The variables $x_1, x_2, \cdots, x_n$ are called the *inputs* of the circuit. Any combination of values of these variables is called an *input combination.* The set of functions f_{ij} of $x_1, x_2, \cdots, x_n$, which are 1 for exactly those states of the input such that t_i and t_j are connected, may be called the *outputs* of the circuit. (We define $f_{ii}=1$ for all i since any terminal may be regarded as always connected to itself.) Any particular set of values of these functions will be called an *output combination.* Thus a combinational circuit is one in which the output combination depends only on the input combination. All the two-terminal circuits previously discussed are special cases of this more general class of circuits.

In many cases, a multi-terminal (or *multiple-output*) combinational circuit may be designed first as a collection of two-terminal networks, after which one attempts to combine these networks so as to have them share contacts so far as possible. Details of the available techniques are given in [2], Chapter 6. Such a multi-terminal circuit may also be studied with the aid of the corresponding "switching matrix" $[f_{ij}]$. (See [15].)

The problems to be stated below apply in large measure to multi-terminal combinational circuits as well as to the two-terminal kind.

6. Problems. Certain problems arise naturally out of the preceding discussion. The most obvious ones relate to the issue of economical design:

(a) Is there a general way of designing a contact network satisfying given requirements and employing a minimum number of contacts? In the case of a two-terminal combinational network, this is equivalent to asking whether there is a corresponding way of defining and computing a "simplest" form for a switching function.

(b) Is there a general way of reducing a switching function to a form corresponding to the *minimum series-parallel* realization?

(c) Is there a general way of designing a minimum circuit when there are certain input combinations which never occur in the use of the circuit?

(d) Is there a general way of determining whether or not a given two-terminal circuit constitutes the minimum-contact realization of the corresponding switching function?

(e) Is there a general way of determining whether the minimum-contact realization of a given switching function is of series-parallel or of bridge type?

One could state other such mathematical problems, but these serve to illustrate the very practical type of question which may be asked. Not only is a minimal or near-minimal circuit neater and apt to be more reliable in operation, but also, when the same circuit is duplicated hundreds of times in an installation, the elimination of a few contacts from it may result in substantial savings.

There is thus an economic motivation as well as an intellectual one for solving these minimization problems.

A significant fact, from a mathematician's point of view, is that attempts to solve these problems have revealed many interesting algebraic and geometric properties of Boolean functions. In the course of these efforts, a wide variety of mathematical tools, such as group theory [5], lattice theory [6], and a geometrical approach [7], [8], [9], have been used. Some of the basic counting problems connected with this theory have been solved, by implication, in [14]. It seems clear that any further progress toward solution of the above listed problems will require the use of even more of the points of view and techniques of modern abstract mathematics.

There are also problems of an entirely different nature involved in the theory of switching. Certain of these arise in the design of large switching systems, involving perhaps hundreds of thousands of contacts. The complexities encountered in the application of the methods discussed in this paper increase at a rate which is at least of the order of 2^{2^n}, where n is the number of circuit variables, so that any hope of rigorous minimization rapidly disappears. In fact, for large systems, the real question is "*What types* of basic switching circuits should be employed, and *how* should these be employed, in order to provide the most economical systems possible to perform the given functions?" This is a problem which is, in large measure, still unsolved, and which will require keen mathematical thinking for its solution. Long since a basic problem in the construction of telephone switching systems, the problem is rapidly becoming important in other connections as well. The digital computer and the automatic factory provide striking examples. An important point is that conclusions valid for systems based on relay circuitry may be invalid for systems based on electronic circuitry, and *vice versa*, because of fundamental differences in the components employed.

7. Conclusion. Since this has been only a very brief discussion of relay switching theory, we have not treated the use of Boolean algebra in the design and analysis of circuits employing electronic components. Nor have we treated the subject of sequential circuits, namely those in which the output combination depends not alone on the input combination, but also on the past history of the circuit. Discussions of these matters will be found in references [2], [10], [11], [12], [13] listed below.

The fact that these topics were not treated here should not be taken to reflect on their importance. Indeed, with a large variety of electronic switching systems already in use or under development, it is apparent that such systems are destined to supersede relay systems in importance. Again, in both computers and in telephone systems, the output combination typically depends on the past history of the circuit as well as on the input combination. Thus sequential circuits are the kind that are ultimately required. Their design is often extremely

difficult, being complicated by the necessity of incorporating in them certain "memory elements" able to take account of the past history of the circuit. Once this task is accomplished, however, the problem reduces to the design of suitable combinational circuits, which are thus seen to be of fundamental importance.

We hope that the introduction to the basic concepts of combinational circuits which has been given here has been sufficiently clear and inviting to encourage the reader to investigate the further literature on his own, for switching theory—or the theory of digital control processes, as it is sometimes called—is steadily growing in importance and has need of able researchers. Indeed, the fact that abstract problems exist at all suggests that it may be possible to create an inclusive, abstract theory of digital control processes largely transcending the particular technologies used to realize the circuits, and largely unifying the many scattered results now in existence. In this possibility lies a real challenge to the mathematician.

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A GENERALIZATION OF JENSEN'S THEOREM ON THE ZEROS OF THE DERIVATIVE OF A POLYNOMIAL

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The object of this note is to present a proof of the following theorem concerning the position of points in the plane of the complex variable:

THEOREM 1. *Let $p(z)$ be a real polynomial not identically constant all of whose real zeros lie in the interval $\alpha \leqq x \leqq \beta$ of the axis of reals, and let γ be a real point not in the interval $\alpha < x < \beta$. Let each conjugate pair z_k, $\bar{z}_k$ of non-real zeros of $p(z)$ have real part in the open interval $\alpha < x < \beta$, and let $\Gamma(z_k)$ denote the circle through z_k and $\bar{z}_k$ tangent to the line γz_k at z_k. Then all non-real zeros of the derivative $p'(z)$ lie in the closed interiors of the circles $\Gamma(z_k)$.*

Theorem 1 admits only finite points γ, but if γ is allowed to become positively or negatively infinite, we obtain as a limiting case of Theorem 1 a well known result:

JENSEN'S THEOREM. *Let $p(z)$ be a real polynomial not identically constant, and consider the circles having as diameters the line segments $z_k \bar{z}_k$ joining conjugate pairs of non-real zeros of $p(z)$. Then all non-real zeros of $p'(z)$ lie in the closed interiors of these circles.*

Jensen's Theorem represents for real polynomials a sharpening of Lucas's Theorem, that *for an arbitrary polynomial not identically constant, the zeros of the derivative lie in the smallest convex polygon containing the zeros of the original polynomial.*

Both Lucas's Theorem and Jensen's Theorem are ordinarily proved* by use of Gauss's Thorem, that *the zeros of the derivative $p'(z)$ of a polynomial $p(z)$ which are not multiple zeros of $p(z)$ are the positions of equilibrium in the field of force due to unit particles situated at the zeros of $p(z)$, where each particle repels with a force equal to the inverse distance.*

We shall use Gauss's Theorem to prove Theorem 1, but it is convenient first to establish a preliminary result:

LEMMA. *Let Γ denote the circle through the points $+i$ and $-i$, tangent at $+i$ to the line joining $+i$ and $\gamma(>0)$. At an arbitrary non-real point $z = x+iy$, $x < \gamma$, interior [exterior] to Γ the force due to unit particles at $+i$ and $-i$ has a non-vanishing component perpendicular to the line γz sensed toward [away from] the axis of reals.*

The force at a point z due to a unit particle at z_0 is in magnitude, direction,

* See for instance J. L. Walsh, The Location of Critical Points of Analytic and Harmonic Functions, Colloquium Publications of the American Mathematical Sociery, vol. 34, 1950.

The Lemma of the present note is a sharpening of Theorem 1 (*loc. cit.*, p. 28).

and sense represented by the vector $1/(\bar{z}-\bar{z}_0)$, so in the field of force of the Lemma the force at z is

$$\frac{2\bar{z}}{\bar{z}^2+1}=\frac{2(x-iy)[(x+iy)^2+1]}{(x^2-y^2+1)^2+(2xy)^2},$$

whose slope is

$$\frac{y(x^2+y^2-1)}{x(x^2+y^2+1)}.$$

The locus of points (x, y) at which the line of action of the force passes through the point $(\gamma, 0)$ is

$$\frac{y}{x-\gamma}=\frac{y(x^2+y^2-1)}{x(x^2+y^2+1)}, \tag{1}$$

and for non-real z (*i.e.*, $y\neq 0$) this locus is

$$\Gamma: \quad x^2+2x/\gamma+y^2-1=0, \qquad \gamma\neq 0$$

namely the circle through $+i$ and $-i$ whose tangent at $+i$ passes through the point γ. The total force at z is zero when and only when z is zero. It is geometrically obvious that at a point $z=x+iy$, $x<\gamma$, interior [exterior] to Γ near $+i$ or $-i$ the total force has a non-vanishing component perpendicular to the line γz sensed toward [away from] the axis of reals. At a non-real point z the force component perpendicular to γz vanishes when and only when z lies on Γ, and this component varies continuously with z, $z\neq\pm i$. Consequently this component is different from zero and sensed toward [away from] the axis of reals at every non-real point $z=x+iy$, $x<\gamma$, interior [exterior] to Γ.

We are now in a position to apply Gauss's Theorem, and to prove Theorem 1. The case that $p(z)$ has only the single root β is trivial and henceforth excluded. By the possibility of interchanging left and right, and by Lucas's Theorem we need consider merely values $z=x+iy$ with $x<\gamma$. At a non-real point $z=x+iy$ with $x<\gamma$ exterior to all the circles $\Gamma(z_k)$, it follows from the Lemma that the force due to each pair z_k, $\bar{z}_k$ of particles has a non-vanishing component perpendicular to the line γz sensed away from the axis of reals; the force at z due to a single particle at a real zero of $p(z)$ to the left of γ likewise has a non-vanishing component perpendicular to γz sensed away from the axis of reals, so the total force at z is not zero, and z cannot be a position of equilibrium. Of course z cannot be a multiple zero of $p(z)$, so z cannot be a zero of $p'(z)$, and Theorem 1 is established.

In the proof of Theorem 1 we have used the Lemma only insofar as it deals with points z exterior to Γ. A similar consideration of points interior to Γ (details are left to the reader) yields the

COROLLARY. *Let $p(z)$ be a real polynomial all of whose real zeros lie in the interval $\gamma \leqq x < +\infty$ of the axis of reals. Suppose each conjugate pair z_k, $\bar{z}_k$ of non-real zeros of $p(z)$ has real part less than γ, and let $\Gamma(z_k)$ denote the circle through z_k and $\bar{z}_k$ tangent to the line γz_k at z_k. Then no non-real point z interior to all the circles $\Gamma(z_k)$ can be a zero of $p'(z)$.*

As an application of both Theorem 1 and the Corollary, we may choose $p(z)$ as real and of form $(z-z_1)^{m_1}(z-\bar{z}_1)^{m_1}(z-z_2)^{m_2}(z-\bar{z}_2)^{m_2}$, $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, $y_1y_2>0$, and choose for instance γ as the intersection (supposed finite) of the line z_1z_2 with the axis of reals. It follows from Theorem 1 that a non-real zero of $p'(z)$ cannot lie exterior to both $\Gamma(z_1)$ and $\Gamma(z_2)$, and follows from the Corollary that such a non-real zero cannot lie interior to both circles $\Gamma(z_1)$ and $\Gamma(z_2)$. Thus every non-real zero of $p'(z)$ other than a multiple zero of $p(z)$ must lie either on both circles or interior to one and exterior to the other.

Some comments are in order regarding zeros of $p'(z)$ *on the circles* $\Gamma(z_k)$ in Theorem 1. If a non-real point z lies interior to no circle $\Gamma(z_k)$ and either z is exterior to at least one such circle or there exist some real zeros of $p(z)$ not at γ, then z cannot be a position of equilibrium, and can be a zero of $p'(z)$ only if it is a multiple zero of $p(z)$. On the other hand, if $p(z)$ has no real zeros other than perhaps at γ, and if all the circles $\Gamma(z_k)$ pass through a non-real point z' interior to the Lucas polygon for $p(z)$, then it may occur that z' is a zero of $p'(z)$.

It is essential in Theorem 1 to exclude a pair z_k, $\bar{z}_k$ whose real part equals γ, for if the real part equals γ there is no proper circle $\Gamma(z_k)$, and it follows from (1) that except on the line $x=\gamma$ the force at every non-real point z due to the particles at z_k and $\bar{z}_k$ has a component perpendicular to the line γz in sense toward the axis of reals. Nevertheless, in the Corollary non-real zeros of $p(z)$ with real part γ may be admitted; no non-real $z=x+iy$ with $x<\gamma$ interior to all the proper circles $\Gamma(z_k)$ can be a zero of $p'(z)$.

In the Corollary, if $p(z)$ has no real zero except perhaps at γ, then a non-real point z interior to the Lucas polygon may conceivably be a zero of $p'(z)$ if it lies on all the circles $\Gamma(z_k)$; but a non-real point exterior to no circle $\Gamma(z_k)$ and interior to at least one such circle cannot be a zero of $p'(z)$ unless it is a multiple zero of $p(z)$.

If $p(z)$ has but one pair of non-real zeros, Theorem 1 and the Corollary combine to determine a lens-shaped region (*loc. cit.*, p. 39, Theorem 1) containing all non-real zeros of $p'(z)$.

Theorem 1 may be complemented by a result on the number of zeros of $p'(z)$, wholly analogous in content and proof to a complement to Jensen's Theorem (*loc. cit.*, p. 31, Theorem 1).

ACCESSORY LINKAGES WHICH HAVE CERTAIN STABILIZING PROPERTIES

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An example of a remarkably versatile linkage is provided by the crossed parallelogram (the Hart cell). In its symmetrical configuration the midpoints of the longer sides are coincident, while the perpendiculars from these coincident points on the shorter sides are collinear, and constitute an axis of symmetry. If the four sides are now divided proportionally from the vertices which are on the same side of this axis, these four points will not only be collinear: but for every deformation any three consecutive points will act as an inversor, with the outermost point at the pole of the inversion.

While these facts are well known, it is perhaps not so generally known that the Hart cell, with simple adaptations, can be made to describe the cissoids, the lemniscates and the limaçons. These curves, with the one exception of the non-symmetrical lemniscatoids, are the inverses of the conics about their vertices, centers and foci respectively.

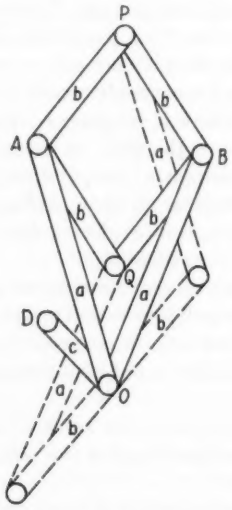

FIG. 1

Peaucellier's cell can also replace Hart's cell and, when it does, stabilization becomes necessary. We shall say a linkage or inversor is *unstable* if its predetermined tracer point is able to depart at any point from its anticipated locus or motion and follow an alternative locus or take up a contrary motion.

It is not the purpose of this paper to develop the equations of the curves mentioned in paragraph two as they appear when developed directly from the

linkage mechanisms. These are adequately dealt with by R. C. Yates in his *Geometrical Tools*, pages 172–185. In what follows, it will be understood that a term such as *elliptic cissoid* will mean that particular form of cissoid whose inverse is an ellipse. With reference to Figure 1, Yates demonstrates that when a^2-b^2 is greater than, equal to, or less than $4c^2$, the point P describes respectively the elliptic, parabolic and hyperbolic cissoids.

If the *dotted* arms a were each extended upward their own lengths and joined at their upper extremities by a bar $2b$ units in length, we would have a Hart cell. The auxiliary link DO from the fixed point D will impart to the point P of the Hart cell cissoidal motion provided the point Q is attached to a fixed pivot. This point Q will be located at the pole of the cissoid, and the whole assemblage will require no stabilization since P and Q (the inversor points) pass and repass each other with perfect smoothness and precision. On the other hand, were the cissoidigraph constructed of the bars represented by the *solid* bars of Figure 1 (*i.e.*, Peaucellier's cell plus DO), P and Q would become unstable in their position of coincidence (at the cusp in the case of the parabolic cissoid and the node in the case of the hyperbolic cissoid). In 1950 the writer stabilized Peaucellier's cell by adding the three bars a, a and $2b$ as shown in the dotted lines. At that time he failed to observe that the addition of exactly one half of a Hart cell was sufficient to stabilize a Peaucellier cell.

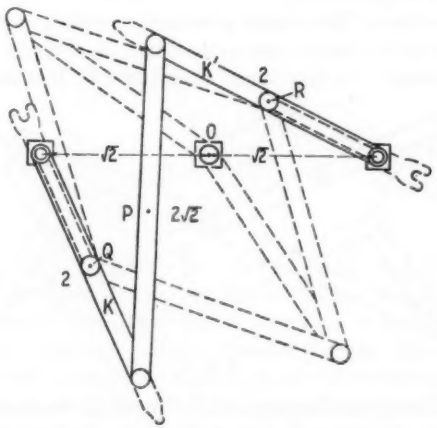

FIG. 2

In 1953, while preparing a linkage demonstration for the Upper New York Section of the Mathematical Association, the writer unwittingly performed a reverse stabilization operation on Bernoulli's lemniscatograph. This instrument is essentially a Hart cell having its opposite vertices pivoted at fixed points. Any point P on the intermediate traversing bar will describe a lemniscatoid and, in particular for the dimensions indicated in Figure 2, the mid-point will de-

scribe the lemniscate of Bernoulli; its inverse with respect to the point O is the equiangular hyperbola having its center also at O.

It is easily seen that the three movable bars become unstable whenever the configuration forces them to lie in the same straight line. The writer is indebted to Michael Goldberg for a non-linkage stabilizer as a means of correcting this situation. Essentially his solution consists of attaching two elliptical gears to the radial arms in such a manner that the extremities of these arms are made coincident with the foci of the pitch ellipses of the gears. The eccentricities of these gears equal the numerical length of a radial arm divided by the numerical length of the traversing arm. It is necessary to use only six strategic teeth from these hypothetical gears to effect stabilization. They are shown in Figure 2.

A true linkage solution to this problem was obtained by making the pivots about which the arms K and K' rotate integral with these links and extending the respective shafts through the supporting plane where they were again made integral with arms just one half the length of the original arms and positioned as shown in Figure 2. To the extremities of these half-length arms the six-bar negative Peaucellier cell was attached with its fixed point joined to O, the point midway between the two rotatable shafts. Now since QOR forms a straight line in both cases and $QO \cdot QR$ equals a constant, the two systems will work together harmoniously and will not require stabilization, since the negative Peaucellier inversor is stable in all positions.

This inverse scriber of the conics possesses complete versatility. Upon altering the dimensions of the radial arms and, of course, those of the half arms at the back of the instrument, a whole range of hyperbolic lemniscates can be drawn

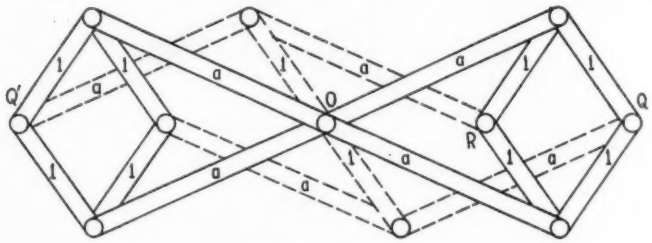

FIG. 3

having their poles at O. Attaching a novel form of Peaucellier's inversor (Fig. 3) to this point, both branches of the hyperbola can be described simultaneously. This last mentioned double inversor was the logical by-product of the stabilized inversor described in connection with Figure 1.

Now while all other intermediate points on the traversing bar of Figure 2 also describe lemniscatoids, the locations of the pole on which the inversor must be set and the center of the resultant conic must be especially calculated. The following simple formulas are quoted without proof for those inversors which generate conics whose vertices are coincident with those lemniscatoidal loops

or vertices which have the larger curvatures. Length b is the length of the shorter of the two radial arms, $2a$ the length of the traversing bar and c the distance of the tracer from the mid-point of this bar. The pole of the inversor is located $(a-c)$ units from that bearing which is more remote from the above mentioned point of osculation and lies on the line joining the two bearings. The focus of the parabola is located $32b^2/(4a^2-5b^2+4)$ units from its vertex. The eccentricity of the conic is $\sqrt{(a^2-c^2)/(a^2-b^2)}$, while the distance between the vertices is given by $4(bc+b^2)/|b-c|$.

The classic among the linkages is the one depicted in heavy lines in Figure 4, and while it has little practical interest as far as the parabola is concerned, it

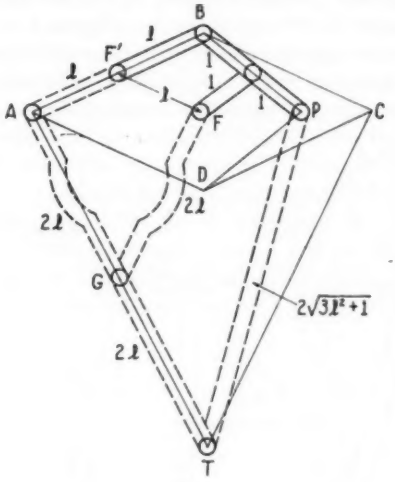

FIG. 4

has the advantage over the linkage just described in the fact that the pole of the inversor as well as the center of the generated conic is invariably found at the fixed point F. Both the points F and F' are fixed in position and are e units apart, where e is the eccentricity of the generated conic. The point P describes the elliptic or hyperbolic lemniscate according as e is less than or greater than 1. Whenever the three points F, F' and P become collinear it will be apparent the linkage will become unstable. The writer was successful in stabilizing this linkage also by means of the following simple method. The left hand link $F'B$ was extended backward its own length e beyond the pivot F' and three additional links were added of lengths $2e$, $4e$ and $2\sqrt{3e^2+1}$ as shown in the dotted lines of Figure 4.

J. J. Sylvester has a simple proof that when two jointed equal-armed V-shaped units (not necessarily congruent) are attached at their free extremities to the opposite angles of a rhombus (a pair of crossed Peaucellier cells), the dis-

tance between the vertices of these V-units is constant. [See Proceedings of the Royal Institute VII (1874).] Accordingly, the calculated distance $2\sqrt{3e^2+1}$ remains constant for every deformation of the double cell. Linkage $ABCDPT$ represents a phantom double cell. It will be observed that the angles BFP and AFT are both right angles with arms lying on the diagonals of $ABCD$. In the case of the elliptic lemniscate it will be found necessary to notch the sections AG and FG at their mid-points. These notches are cut from the inside to enable them to encircle the pivot F'. Now if this pivot were replaced by a stout pin carrying at its upper extremity a stationary arm of length e and having a hole at its outer extremity positioned immediately above F, it would be possible to attach the center O of the double Peaucellier inversor (Fig. 3) to this hole. This inversor is dynamically balanced so that if the point P of Figure 4 were joined to an inner vertex R of this cell, the outer-vertices Q and Q' would describe simultaneously either the two finite branches of the ellipse, drawing it in its entirety, or the two branches of the hyperbola each passing through a vertex.

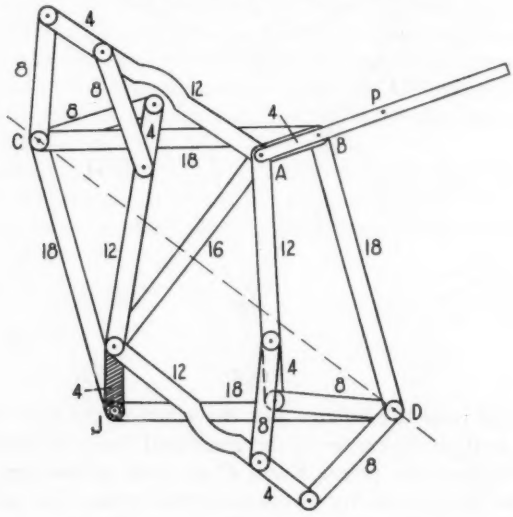

FIG. 5

Finally we come to a description of the limaçonigraph. A single auxiliary bar will convert a Hart cell into a linkage capable of drawing the hyperbolic and elliptic limaçons including the cardioid (parabolic limaçon). This versatile linkage is found described in *Geometrical Tools*. Despite the fact that the writer succeeded in also finding a known stabilizing linkage for this classic linkage, he feels it might be more interesting to describe one of his own design. The actual model works with great smoothness and precision. In the diagram of the instrument (Fig. 5) the short shaded bar 4 is attached rigidly to a stout dowel pin J.

This pin passes through the supporting base and is integral with it. The corresponding short bar AB is cranked outwardly through bar 16, which connects the shaded bar with the bar AB. It also goes through a leg belonging to each of two Hart straight line scribers, which line is the perpendicular bisector of the bar 16.

On top of the shaft which goes through these three parts is a rigidly positioned crank parallel to AB and pointing in the same direction. This latter part is the uppermost of all the links and carries the scriber, which can be set at variable distances from A. To visualize the action of the linkage let us suppose that bar 16 is held fixed instead of the shaded bar 4. It can then be easily seen that as the shaded bar is rotated, say clockwise with uniform angular velocity, the bar AB would rotate counter-clockwise with uniform velocity also, since vertex C and vertex D of the rhombus of sides 18 are constrained to move along the diagonal due to the two "A-Linkages" (Hart's) which are positioned on the common base 16. Now giving the shaded bar angular velocity zero as in the original set-up, it is apparent that the angular velocities of bars 16 and AB relative to this fixed bar are in the exact ratio of 1:2 and in the same sense.

The setting of the adjustable tracer P governs the character of the limaçon. When the tracer is set at 8 units from A it describes the cardioid, at lesser distances the elliptic limaçon (loopless), and at a greater distance than 8 from A, the hyperbolic limaçon. It is to be hoped that the principles set forth in the foregoing may be of value in mechanical situations which require the elimination of dead-centers.

A CARD GAME WITH BLUFFING

RUFUS ISAACS, The Rand Corporation

The present game—named by our daughter "Guess It"—has the following features:

A play involves not a paucity of moves but a sequence, possibly of considerable length.

It is a game of incomplete information with a kind of bluffing possible for almost all positions.

Unlike most of the examples in expositions of game theory, it has merit as a recreation. At least we have been so told by persons who have played it. The decisions as to bluffing and bluff detection appear stimulating especially to those with a flair for gambling, psychological games.

It is a rare thing for a game with such properties to be amenable to analysis. This one is; fortunately the information structure that a player acquires as he

plays is not cumulative. That is, positions recur which are tantamount to starting positions and we can solve the game by a recurrence or "programming" technique.

The optimal strategies are, of course, mixed.* Such have been the subject in recent years of much lucubration and little action. Remedially, we furnish two dials. The reader can manufacture spinners for their centers and thus possess the chance device needed for actual ventures into optimal playing.

The game. A deck of consecutively numbered cards is randomly dealt to two players, but one card is placed face down on the table. (The objective is to identify this hidden card.) For his move, a player may either *call* or *ask*.

If he calls, he announces a possible name for the hidden card. The game is now over. If correct, the caller wins; if not, he loses.

If he asks, he inquires of his opponent, "Have you such-and-such a card?" and the opponent must reply truthfully. It is then the other player's turn to move; he is faced with the same two choices. The same card may not be asked about twice.

Thus the art of the game is to acquire information with as little disbursement of it as possible until we have enough knowledge—or margin thereof—of the hidden card to warrant a call.

When we speak of a player bluffing we mean that he is asking about a card in his own hand. If a player decides never to bluff, and so informs his opponent, it is clear that, if he happens to ask about the hidden card, he loses the game. Some bluffing is thus essential for defense; offensively it may be used to decoy the opponent into a false call.

The various positions in a play differ only in their information structure. But from an alternative viewpoint we may regard the difference as essentially a tangible one with most of the residual information nugatory. Consider, for example, the opening move.

If it is a call, the play is over.

If it is an ask with a "yes" reply, then both players know the location of one card. From now on it cannot be asked for and will not be called. Thus nothing is changed if we think of this card as deleted. The original type of position has recurred with one card less and a change in turn. (In actual play, such effectively deleted cards may be placed face up on the table.)

For the remaining alternative suppose that Player I asks Player II about card x and is answered "no." The latter player is now in a quandary about card x, but his dilemma will not long persist. If he does not call, and if x was the hidden card, Player I most certainly will. Hence if Player I asks on his next turn, he must have x in his hand. Both players know it, and it may be deleted.

* A player employs a *mixed* strategy when he selects his alternatives at each move with certain probabilities rather than making definite decisions. This concept, due to J. von Neumann, is expounded in any text on game theory.

The analysis. We operate under the following assumptions, which eliminate some absurd, dominated strategies.

(1) If a player knows the hidden card with certainty and has the opportunity, he will call and win.

(2) A player never will call a card he knows is not the hidden one.

(3) A player will make a choice from a set of indifferent cards equiprobably. For example, if he holds ten cards and on his first move has decided to bluff, he will select each of his cards with a probability of one-tenth.

Let $P(m, n)$ be the win probability of the player with first move if, at the opening of a game, he holds m cards and his opponent n, under the assumption that both players will play optimally. Then $P(m, n)$ is the value of the game if we choose the payoff as 1 and 0 according as the first move player wins or loses.

Let $A(m, n)$ be same quantity, except that now the rules have suffered the amendment: the opening move must be an ask.

Inasmuch as a first move call has a probability of success equal to $1/(n+1)$, we have

$$P(m, n) = \max\left[\frac{1}{n+1},\ A(m, n)\right]. \tag{1}$$

Assumption (1) shows that

$$P(m, 0) = 1,$$

and thus the half-a-loaf adage, that

$$P(0, n) = \frac{1}{n+1}. \tag{2}$$

Also

$$A(m, 0) = \frac{m}{m+1}, \tag{3}$$

for a player can always ask without betraying information by choosing his card equiprobably from the entire deck (a policy hereafter referred to as the *safe move*). The first player with a monopoly of cards and under compulsion to ask, should act so, thus putting his opponent in the position governed by (2).

We turn to mixed strategies, supposing now $m>0$, $n>0$. Let the first move —made by, say, Player I—be an ask and let b be the probability that he shall bluff. Let Player II's strategy be in part:

If the reply is "no" he will with probability

c_1, call the card x that was asked about.	(α)
c_2, call some other card.	(β)
$1-c_1-c_2$, ask.	(γ)

Let us suppose I bluffs and see what happens under these three responses.

(α) Player I wins; hence the outcome has the value 1.

(β) Player II has probability $1/m$ of calling correctly; hence the outcome is $(m-1)/m$.

(γ) If Player II asks he must make his decisions as to bluffing on the premise that card x lies in Player I's hand. For if x were the hidden card, Player I would know it and so call on his move; in this case the nature of II's ask would be immaterial. In the contrary case—where the nature of the ask is effective—II's premise will be verified.

Now if we hypothesize that I is bluffing, card x will be on the verge of deletion. As we have just seen, II will act as if it already were and with I's response it will be. Thus the situation is as if II had the opening move and n cards and I had $m-1$ cards. The value is then

$$1 - A(n, m-1).$$

Now suppose I had not bluffed but has received a "no." He has hit upon the hidden card; the probability for this situation is $(1-b)/(n-1)$. The three outcomes are:

(α) II wins; outcome $=0$.

(β) II loses; outcome $=1$.

(γ) As I will call on the next move and win, the outcome $=1$.

The probability that I receives a "yes" reply is $(1-b)\cdot n/(n+1)$. The outcome, being tantamount to a fresh start, is $1-P(n-1, m)$. We may then write

$$A(m, n) = \max_{b} \min_{c_1, c_2} \left\{ b\left[c_1 + c_2 \frac{m-1}{m} + (1 - c_1 - c_2)(1 - A') \right] + \frac{1-b}{n+1}[1 - c_1] + \frac{n(1-b)}{n+1}(1 - P') \right\} \quad (m > 0, n > 0), \tag{4}$$

where $A' = A(n, m-1)$, $P' = P(n-1, m)$.

We should note that in (4) the order of taking the max and min is immaterial. This follows from the central theorem of game theory if we can show that the right side of (4) is derivable from the pay-off matrix of some game. Such a game is obtained from ours through the following modifications:

Assumptions (1) and (2) are embodied as rules. In place of assumption (3) we put corresponding equiprobable chance moves. At the end of one round of moves the players are paid off, A' and P' being taken as any prescribed numbers.

The reader can easily verify that (1), (2), (3), (4) determine uniquely $P(m, n)$ for all m, n.

As (c_1, c_2) is restricted to the triangle $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 \leqq 1$ and the brace of (4) is linear in c_1, c_2, its minimum will be assumed at one of the vertices of the triangle. Thus

$$A(m, n) = \max_{b} \min_{i} \left[\frac{n}{n+1}(1 - P') + Z_i \right], \qquad (i = 0, 1, 2),$$

where

$$Z_1 = b\left(1 - \frac{n}{n+1}(1 - P')\right) = b\left(\frac{1}{n+1} + \frac{n}{n+1}P'\right), \qquad (c_1 = 1, c_2 = 0),$$

$$Z_2 = b\left(\frac{m-1}{m} - \frac{1}{n+1} - \frac{n}{n+1}(1 - P')\right) + \frac{1}{n+1}$$

$$= b\left(-\frac{1}{m} + \frac{n}{n+1}P'\right) + \frac{1}{n+1} \qquad (c_1 = 0, c_2 = 1),$$

$$Z_0 = b\left(1 - A' - \frac{1}{n+1} - \frac{n}{n+1}(1 - P') + \frac{1}{n+1}\right)$$

$$= b\left(-A' + \frac{n}{n+1}P'\right) + \frac{1}{n+1}, \qquad (c_1 = c_2 = 0).$$

We evaluate the minimax in two cases only. Then we will prove that no other possibilities can arise.

Case 1. $m = 1$.

Here $A' = A(n, 0) = n/(n+1) < 1 = 1/m$. Thus $Z_0 > Z_2$ and we need compare only Z_1 and Z_2. The coefficient of b is positive in the former and negative in the latter. Then clearly the maximum occurs when $Z_1 = Z_2$ or when

$$b = \frac{1}{n+2} \text{ (the safe move!).} \tag{5}$$

Then, after a short reduction,

$$A(1, n) = \frac{1 + n(1 - P')}{n+2}. \tag{6}$$

Case 2.

$$A' \geqq \frac{1}{m}, \qquad A' \geqq \frac{n}{n+1}P'.$$

Here $Z_0 \leqq Z_2$ so that the race is between Z_0 and Z_1. The former having a negative coefficient of b, the maximum again occurs at the equality point. Thus brief calculations give for the maximum:

$$b = \frac{1}{1 + (n+1)A'} \tag{7}$$

and

$$A(m, n) = \frac{1 + nA'(1 - P')}{1 + (n+1)A'}. \tag{8}$$

LEMMA 1. *If* $m>0$, $n>0$, *then*

$$A(m, n) \geqq \frac{1}{n+1}, \tag{9}$$

and consequently $A(m, n)=P(m, n)$.

Proof:* Let Player I adopt the strategy: his first move is safe; if II does not call in response, I does so on his next turn. Then if II calls his probability of correctness $\leqq 1/(m+1)$ and I attains at least $m/(m+1) \geqq \frac{1}{2} \geqq 1/(n+1)$. If I calls, his correctness probability also $\geqq 1/(n+1)$.

LEMMA 2. *If* $m>0$, $n>0$,

$$A(m, n) \geqq P(m-1, n+1). \tag{10}$$

Remark. These two lemmas ensure Case 2 if $m>1$. The latter does so more strongly than necessary, for it shows

$$A' \geqq P' > \frac{n}{n+1} P'.$$

Proof: We work inductively on $k=m+n$. Its smallest value is 2 and here

$$A(1, 1) = \frac{1+(1-P(0, 1))}{1+2} = \frac{1+\frac{1}{2}}{3} = \frac{1}{2} > \frac{1}{3} = P(0, 2).$$

We now assume $k>2$ and

(11) (10) holds for $m+n<k$, $m>0$, $n>0$.

(a) $m=1$. From (6) and (2)

$$A(1, n) = \frac{1+n(1-P')}{n+2} \geqq \frac{1}{n+2} = P(0, n+1).$$

(b) $m=2$. We may apply (8) for, as $A'=A(n, m-1)$, (9) and (11) assure us of Case 2. Thus

$$A(m, n) = \frac{1+nA'(1-P')}{1+(n+1)A'} \geqq \frac{1+nA'(1-A')}{1+(n+1)A'} \tag{12}$$

while

$$P(1, n+1) = \frac{1+(n+1)(1-P(n, 1))}{n+3} = \frac{1+(n+1)(1-A')}{n+3}. \tag{13}$$

We attain our result by forming the difference of the rightmost terms of (12) and (13). The numerator turns out to be

* Suggested by S. Karlin.

$$1 + (n - 1)A'(1 - A') > 0.$$

(c) $m > 2$. Now (12) stands, but (13) is to be replaced by an instance of (8):

$$P(m - 1, n + 1) = \frac{1 + (n + 1)A''(1 - P'')}{1 + (n + 2)A''}, \tag{14}$$

where $A'' = A(n+1, m-2)$, (9) and (10) applying for these arguments, and $P'' = P(n, m-1) = A'$.

A subtraction of the right sides of (14) and (12) yields for the numerator, after some rearrangement,

$$[A'' - A'] + n[A''P'' - A'P'] + A''[P'' - A'] + A'A''[(n + 1)^2P'' - n(n + 2)P'].$$

This expression cannot be negative as (11) implies

$$A'' \geqq P'' = A' \geqq P'.$$

We may now eliminate the symbol A from our results and embody them in a

THEOREM. *The function $P(m, n)$ satisfies and is determined by*

$$P(m, 0) = 1, \qquad P(0, n) = \frac{1}{n + 1}$$

$$P(1, n) = \frac{1 + n(1 - P(n - 1, 1))}{n + 2} \qquad (n \geqq 1),$$

$$P(m, 1) = \frac{1 + P(1, m - 1)\dfrac{m}{m + 1}}{1 + 2P(1, m - 1)} \qquad (m \geqq 1),$$

$$P(m, n) = \frac{1 + nP(n, m - 1)[1 - P(n - 1, m)]}{1 + (n + 1)P(n, m - 1)}, \qquad (m > 1, n > 1).$$

We shall append a brief table of this function.

The optimal strategies for a player asking are given by (5) and (7). We have still to calculate the best values of c_1, c_2.

Let us return to (4) and rewrite the brace:

$$b\left[c_1\left(A' + \frac{1}{n+1}\right) + c_2\left(A' - \frac{1}{m}\right) - \left(A' - \frac{n}{n+1}P'\right)\right] + 1 - \frac{n}{n+1}P' - \frac{c_1}{n+1}. \tag{15}$$

In both our cases the optimal b is between 0 and 1. It follows that the optimal c_1, c_2 must be such as to annul the bracket in (15). Thus in the (c_1, c_2)-plane, we are confined to a line and also to the triangular set: $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 \leqq 1$. A

glance at the remaining terms of (15) shows that we must select the point on the intersection of the line and triangle with the largest possible c_1. Let the equation of the line be

$$a_1c_1 + a_2c_2 = a_3. \tag{16}$$

Case 1. Here $a_1>0$, $a_2<0$ and, as

$$A' = A(n, 0) = \frac{n}{n+1},$$

$a_3>0$. The line thus has positive slope and meets the c_1-axis at a_3/a_1, which is between 0 and 1. Thus the sought point is on the hypotenuse

$$c_1 + c_2 = 1$$

which must be solved with (16). In this case the latter is

$$c_1 - \frac{1}{n+1}c_2 = \frac{n}{n+1}(1 - P'),$$

and the solution is

$$c_1 = \frac{(n+1) - nP'}{n+2}$$

$$c_2 = \frac{1 + nP'}{n+2}, \qquad (P' = P(n-1, 1)). \tag{17}$$

Case 2. Now $a_1>0$, $a_2>0$, $a_3>0$. The line has negative slope and, as before, meets the c_1-axis at a point between (0, 0) and (1, 0). Thus

$$c_1 = \frac{(n+1)A' - nP'}{1 + (n+1)A'}$$

$$c_2 = 0. \tag{18}$$

Description of a typical play. We suppose optimal strategy on both sides. Suppose first that Player II has but one card.

The play will expire quickly. Let Player I ask. If the reply is "yes," he has ascertained the hidden card; II calls in self-defense. If "no," (18) shows that II should likewise call.

Now let m and n be both large and fairly close to equality. It may be expected that P will be somewhere near $\frac{1}{2}$. At the opening position or any other where deletions have cleared the air of information, there will be an ask. The bluff probability being roughly (see (7)) $2/(n+3)$ and thus small, most moves will entail "yes" answers and so alternate deletions from each hand. A chance hit on the hidden card will or a bluff may terminate the play abruptly at any time. If such does not happen the pattern of alternate deletions will be disturbed

by an occasional detected bluff which will remove a card from the *asker's* hand. Thus there will be a random deviation from equality of hands even if m equaled n at the outset. Things come to an end, as we have seen, when one player is reduced to a single card.

The Dials. The b dial instructs Player I (whose move it is) when to bluff. There is a circle for each value of m with marks on it to show n.* The player should bluff when the pointer lands in the sector extending from the appropriate mark clockwise to the horizontal line.

The c_1 dial is used similarly by Player II when his response is "no." He calls the card just asked when the pointer falls in the sector from mark to line. When it does not, he calls the other unknown card when $m=1$ and asks if $m>1$. He should note that on *this* dial, m is the number of cards in his *opponent's* hand. If on his preceding move he bluffed (and the opponent did not call) he should consider the bluffed card as deleted when reckoning n.

The largest deck for which the dials can suffice contains 11 cards.

Some variations. There are numerous possibilities of deriving games, interesting for recreation if difficult for analysis, by altering the rules of the present one.

There might be several hidden cards. The deck may be divided into suits with one hidden card from each. An ask may concern several cards, but a full response is not obligatory. It may state, for example, that one of the cards is held but not specify which; there is no deletion here, of course. The response may be a full reply about any one asked card in the responder's hand; deletion is restored.

Stimulating variants arise when there are more than two players.† The rules should specify whom a player may ask—such as his left neighbor, his choice, or the outcome of a chance move. The rules also declare how much of the query or reply is privy to the parties concerned and how much is made public. The novelty is the possibility of deductive inference from this partial knowledge of the asks and replies of other players. It should be possible to devise rules with a great gamut of scope for ingenuity.

A TABLE OF $P(m, n)$

$P(m, n)$		m				
		1	2	3	4	5
n	1	.5	.667	.688	.733	.75
	2	.5	.556	.625	.648	.680
	3	.4	.512	.548	.597	.619
	4	.375	.450	.513	.543	.581
	5	.333	.423	.467	.512	.538

* Of course m and n are the number of remaining cards after the proper deletions.

† We advance no ideas as to the meaning of an optimal strategy in these cases.

m
1
2
3
4
5
1,2
1
2
3
4
5
1
2
3
4
5
1
2
3
4
5

MATHEMATICAL NOTES

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NOTE ON UNIVALENT FUNCTIONS

VIKTORS LINIS, University of Saskatchewan and University of Ottawa

Friedman [2] has proved the following theorem which is a case of Salem's theorem on univalent functions:

Let $f(z)$ be analytic and univalent in the unit circle $|z|<1$ and let its expansion be $z+\sum_{n=2}^{\infty}a_nz^n$. If all the coefficients a_n are rational integers then $f(z)$ is a rational function and one of the forms:

$$z,\quad z/(1\pm z),\quad z/(1\pm z^2),\quad z/(1\pm z)^2,\quad z/(1\pm z+z^2).$$

The purpose of this note is to give a short proof of this theorem. Let

$$z/f(z)=\sum_{n=0}^{\infty}b_nz^n; \tag{1}$$

then following Prawitz [3] we have

$$\sum_{n=1}^{\infty}(n-1)\,|b_n|^2\leqq 1. \tag{2}$$

Since $b_0=1$, $b_1=-a_2$ and $|a_2|\leqq 2$ (Nehari [1]) it follows that $|b_1|\leqq 2$ and from (2) $|b_n|\leqq 1$ for $n\geqq 2$. The coefficients $b_n(n\geqq 1)$ can be computed from the relation

$$(-1)^nb_n=\begin{vmatrix} a_2 & 1 & \cdots & 0\\ a_3 & a_2 & 1\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ a_{n+1} & a_n & \cdots & a_2\end{vmatrix}$$

and thus are rational integers. The possible values for b_n are:

$$b_0=1;\quad b_1=0,\pm 1,\pm 2;\quad b_2=0,\pm 1;\quad b_n=0\quad \text{for } n\geqq 3.$$

From the combination of these values we obtain 15 functions; however, the following 6 must be rejected as having zeros in the unit circle:

$$1\pm 2z,\qquad 1\pm 2z-z^2,\qquad 1\pm z-z^2.$$

The remaining 9 functions prove the theorem.

Remark. The theorem remains true if the "rational integers" are replaced by the "Gaussian integers." The possible values for b_n are:

$b_0 = 1$; $b_1 = 0, \pm i, \pm 1, 1 \pm i, -1 \pm i, \pm 2i, \pm 2$; $b_2 = 0, \pm i, \pm 1$;
$b_n = 0$ for $n \geqq 3$.

From the 65 possible combinations of these values only 15 satisfy the conditions of the theorem. In addition to the 9 functions listed above we obtain also the following:

$$z/(1 \pm iz), \qquad z/(1 \pm iz)^2, \qquad z/(1 \pm iz - z^2).$$

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A PROOF OF MORLEY'S THEOREM

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Morley's Theorem arose as a byproduct of some deeper geometrical results which F. Morley [1] obtained about fifty years ago. It can be stated as follows: *The points of intersection of the adjacent trisectors of the interior angles of a triangle form the vertices of an equilateral triangle.* Most of the many different proofs of this theorem which have appeared are either geometrical [2] or trigonometrical [3]. The procedure followed here, on the other hand, finds the coordinates of the points of intersection of the angle trisectors as complex numbers [4].

To carry out the proof we refer to the figure, where ABC may be any triangle. Draw the circumscribed circle of ABC, denote its radius by r, let the origin be at the center of the circle, and let the positive real axis pass through the vertex C of the triangle. Let the vertices A and B be represented by the complex numbers $re^{i\alpha}$ and $re^{i\beta}$, respectively, where α and β are positive and less than 2π. Then the trisectors of the interior angles of the triangle will intersect the circle in the points with the complex numbers indicated in the figure.

In order to find the points of intersection of the angle trisectors, we use the following result [5], which can be established readily: *If two intersecting lines cut the circle $|z| = r$ in the points given by the complex numbers a, b and c, d, respectively, the point of intersection of the lines is given by*

$$z = (a^{-1} + b^{-1} - c^{-1} - d^{-1})/(a^{-1}b^{-1} - c^{-1}d^{-1}).$$

Application of this formula to the two adjacent trisectors intersecting at N gives for N the complex number

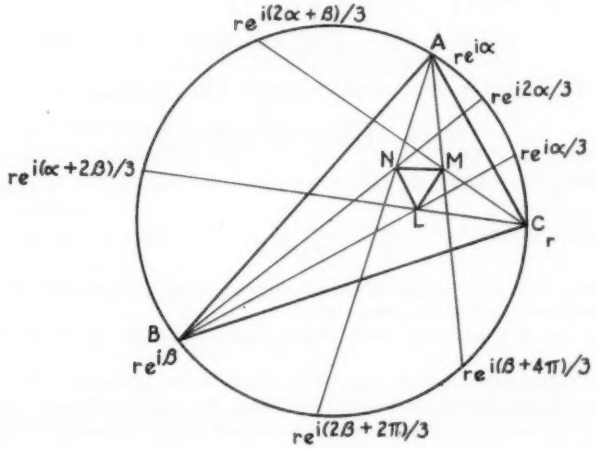
$$z_N = r\,\frac{e^{-i\beta} + e^{-i2\alpha/3} - e^{-i(2\beta+2\pi)/3} - e^{-i\alpha}}{e^{-i(\beta+2\alpha/3)} - e^{-i[\alpha+(2\beta+2\pi)/3]}}.$$

With the notation

$$p = e^{i\alpha/3}, \qquad q = e^{i\beta/3}, \qquad \epsilon = e^{i\pi/3},$$

the expression for z_N is easily reduced to

$$z_N = r(\epsilon p^2 q - \epsilon^2 q^2 p + p^2 - pq\epsilon + q^2\epsilon^2).$$



Similarly, for the points of intersection L and M, we find

$$z_L = r(q^2 p - q^2 + qp - q + p)$$

and

$$z_M = r(p^2 q + \epsilon^2 pq - \epsilon^2 p^2 + p\epsilon - \epsilon q).$$

The vector $\overrightarrow{LN}$ is given by the complex number

$$z_N - z_L = r[\epsilon p^2 q - (\epsilon^2 + 1)q^2 p + p^2 + (\epsilon^2 + 1)q^2 - pq(\epsilon + 1) + q - p]$$

which can be factored and written as

$$\overrightarrow{LN} = -r(q - p)(\epsilon^2 - q)(1 - p)\epsilon.$$

In a similar way

$$\overrightarrow{LM} = -r(q - p)(\epsilon^2 - q)(1 - p),$$

$$\overrightarrow{MN} = -r(q - p)(\epsilon^2 - q)(1 - p)\epsilon^2.$$

Consequently

$$|\overrightarrow{LM}| = |\overrightarrow{MN}| = |\overrightarrow{NL}|,$$

and we conclude that the triangle LMN is equilateral.

The length of side of this equilateral triangle is given by

$$\begin{aligned}|\overrightarrow{LN}| &= r\,|q - p|\,|\epsilon^2 - q|\,|1 - p| \\ &= r \cdot 2 \sin [(\beta - \alpha)/6]\, 2 \sin [(2\pi - \beta)/6]\, 2 \sin (\alpha/6).\end{aligned}$$

In terms of the angles A, B, C of the triangle ABC, this becomes

$$8r \sin (A/3) \sin (B/3) \sin (C/3),$$

a formula given by Kowaleski [3].

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FREDHOLM INTEGRAL EQUATIONS, THE RECIPROCALS OF WHOSE SOLUTIONS ARE ALSO SOLUTIONS

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We wish to prove the following:

THEOREM. *Let C be a bounded measurable set in n dimensional Euclidean space. Let $K(x, y)$ be a real-valued function in $L_2(C, C)$ and consider the Fredholm integral equation:*

$$u(x) = \int_C K(x, y)u(y)dy. \tag{1}$$

Suppose that this equation has the following properties:

(2) *There exists a real solution $u_0(x)$ of (1) in $L_2(C)$ whose reciprocal is in $L_2(C)$;*

(3) *If the reciprocal of a real solution $u(x)$ of (1) in $L_2(C)$ is in $L_2(C)$, then it is itself a solution;*

(4) *If $u(x)$ is a real solution of (1) in $L_2(C)$, then there exists a constant b such that $[b+u(x)]^{-1}$ is in $L_2(C)$.*

Then there is a positive integer m and a decomposition

$$C = \sum_{i=0}^{m} C_i \tag{5}$$

of C into measurable sets C_j such that C_0 and C_iC_j have measure zero if $i \neq j$, and

$$\int_C K(x, y)\chi(y, C_j)dy = \chi(x, C_j) \qquad (j \neq 0), \tag{6}$$

in which the characteristic functions

$$\chi(x, C_j) = \begin{cases} 1 & \text{on } C_j \\ 0 & \text{elsewhere} \end{cases}$$

are a complete set of linearly independent solutions of (1).

Proof. If $k>0$, then $u_0+ku_0^{-1}$ is a real solution whose reciprocal $u_0/(k+u_0^2)$ is bounded by $2^{-1}k^{-1/2}$ and is therefore a solution of (1). Since $K(x, y)$ is in $L_2(C, C)$, equation (1) has only a finite number m of linearly independent solutions [1]. Select $m+1$ distinct positive numbers $k_i (i=0, 1, \cdots, m)$. Then the solutions $u_0/(k_i+u_0^2)$ are linearly dependent, so that there exist constants a_i not all zero and a set E_0 of measure zero such that

$$\sum_{i=0}^{m} \frac{a_i}{k_i + u_0^2} = 0 \qquad (x \text{ not in } E_0). \tag{7}$$

Since the numbers k_i are distinct and the numbers a_i are not all zero, this is an equation in u_0^2 which has at most m solutions, and so u_0 takes on only a finite number of distinct values α_h on $C-E_0$:

$$u_0 = \sum_{h=1}^{r} \alpha_h \chi(x, E_h) \qquad (r \leqq 2m), \tag{8}$$

in which the sets E_h are measurable subsets of C whose sum is $C-E_0$ and whose pairwise intersections are of measure zero.

The assertion that $u_0/(k+u_0^2)$ is a solution of (1) now becomes

$$\sum_{h=1}^{r} \frac{\alpha_h}{k+\alpha_h^2} \int_{E_h} K(x, y)dy = \frac{\alpha_j}{k+\alpha_j^2} \qquad (x \text{ in } E_j). \tag{9}$$

Since this identity holds for all positive k,

$$\int_{E_h} K(x, y)dy = \delta_{hj} \qquad (x \text{ in } E_j), \tag{10}$$

so that the functions $\chi(x, E_h)$ are linearly independent solutions of (1). Therefore $r \leqq m$, and

$$1 = \sum_{h=1}^{r} \chi(x, E_h) \tag{11}$$

is a solution of (1).

Let $u(x)$ be any solution of (1) in $L_2(C)$. Then by virtue of (11), $b+u$ is also a solution of (1), and by hypotheses (3) and (4), there is a value b for which $(b+u)^{-1}$ is a solution of (1). Repeating the argument of the last paragraphs with u_0 replaced by $b+u$, we infer that there are distinct values β_j and measurable sets $F_j (j=1, \cdots, s \leqq m)$ such that

$$u = \sum_{j=1}^{s} \beta_j \chi(x, F_j) \qquad (x \text{ in } C - F_0), \tag{12}$$

$$\int_C K(x, y)\chi(y, F_j)dy = \chi(x, F_j) \qquad (j \neq 0), \tag{13}$$

$$C = F_0 + \sum_{j=1}^{s} F_j, \quad \text{meas } F_0 = \text{meas } F_h F_j = 0 \qquad \text{if } h \neq j. \tag{14}$$

If the number r of sets E_h associated with u_0 is equal to m, the proof of our result is complete. If $r<m$, there is a solution $u(x)$ of (1) in $L_2(C)$ which is linearly independent of the functions $\chi(x, E_h)$. This solution may be represented as in equation (12). Pick numbers α_h $(h=1, \cdots, r)$ so that the values $\alpha_h+\beta_j$ are all distinct, and consider

$$\begin{aligned} U &= \sum \alpha_h \chi(x, E_h) + \sum \beta_j \chi(x, F_j) \\ &= \sum (\alpha_h + \beta_j)\chi(x, E_h F_j). \end{aligned}$$

The function U is a solution of (1) in $L_2(C)$ which is constant on each of the disjoint sets $E_h F_j$ of the partition of $(C-E_0)(C-F_0)$. At least $r+1$ of these sets

have positive measure, for otherwise $u(x)$ would be linearly dependent on the functions $\chi(x, E_k)$. The characteristic functions $\chi(x, E_kF_j)$ of those sets E_kF_j of positive measure can then be shown to be linearly independent solutions of (1) by repeating the argument of the above three paragraphs. In this manner we increase the number of sets until we have a decomposition into exactly $m+1$ sets as prescribed in equation (5).

Since the function $u_0/(k+u_0^2)$ is bounded when $k>0$, it is clear that our argument is equally valid for integral equations satisfying a modified version of (2), (3) and (4) in which $L_2(C)$ is replaced the second time it occurs by any linear subset of $L_2(C)$ which contains the set $M(C)$ of bounded measurable functions on C.

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CLASSROOM NOTES

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ON PROVING THE CHAIN RULE

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Choose an introductory textbook in calculus at random and examine its proof of the chain rule for differentiating a composite function. The proof will fall into one of three categories according to the amount of care taken, but regardless of the category the proof will be found to be less than satisfactory for the beginning student.

Assuming y to be a differentiable function of u, and u a differentiable function of x, a proof in what we may call category A consists in writing the identity $\Delta y/\Delta x=(\Delta y/\Delta u)(\Delta u/\Delta x)$, letting Δx approach zero, noting that Δu then approaches zero, and inferring that $dy/dx=(dy/du)(du/dx)$. This proof, of course, is defective in that it overlooks the possibility that, regardless of the restriction placed on Δx, Δu in approaching zero may attain the value zero "on the way," in which case $\Delta y/\Delta u$ cannot approach dy/du.

A proof in category B is like the above, but contains the additional remark that Δu must not be zero. Since this remark is always made with the utmost conciseness, often in a brief footnote, unaccompanied by any kind of clarifying discussion, this proof can be regarded as an improvement over the preceding only

if it is granted that it is better to leave a student mystified by a difficulty than ignorant of it.

The proofs in category C are rigorous and complete, disposing of the difficulty noted above by an ϵ-procedure of some sort. However, it is hard to see how these comparatively sophisticated proofs can be understood by any but a very small minority of students when it is realized that the latter are usually in contact with the calculus for only about a month when the chain rule is presented to them.

How, then, should we prove the rule to such students? Admittedly, the proof cited above is delightfully simple in outline. That it has not been presented better in our introductory textbooks must be charged, it seems, to the paralyzing force of tradition, *i.e.*, what is good enough for fifty calculus books is good enough for the fifty-first. The fact is, however, that this proof can easily be made rigorous and also understandable, provided that we are willing, as we should be, to make a slight sacrifice in the generality of the theorem. But some simple preliminaries are necessary. First, examples should be given to show that when $u(x)$ is an elementary function of the type familiar to the student Δu can be zero when Δx is not zero, but it cannot be zero if Δx is sufficiently small. Second, it should be shown graphically that $u(x)$ would have to be a most unusual type of function (unless it is a constant) if Δu could be zero regardless how small Δx may be. Third, it should be shown why Δu must not be permitted to equal zero if the limit of $\Delta y/\Delta u$ is to exist and equal dy/du.

With these preliminaries out of the way we can proceed to the proof. If x_1 is the value of x under consideration, and u_1 the corresponding value of u, all that is now necessary beyond what is presented in textbooks is to agree to consider only functions $u(x)$ for which Δu is not zero if x is sufficiently close to x_1 but unequal to it. That is, we state and prove the chain rule only for such functions. Thus, we only exclude functions taking on the value u_1 infinitely many times in the neighborhood of x_1. Except for the trivial case when $u(x)$ is constant in this neighborhood, such excluded functions never occur in first year work, and but rarely in the following few years. The student should be told that the chain rule also applies to these functions but that the proof of this fact is best delayed until such time as he works with them.

BRIEF REMARK ON TEACHING DIFFERENCE EQUATIONS

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The student of difference equations who has studied differential equations notices the many analogous features of the two. However, the substitutions used in solving for the complementary functions of certain equations, namely, of e^{mx} in one case and of a^x in the other, may *look* different. If the student is not fully convinced of the similarity in the procedures by the statement that $e^{mx} = (e^m)^x = a^x$, then go through the steps showing how the substitution of a^x can be used in solving differential equations and e^{mx} in solving difference equations.

A USEFUL REDUCTION INTEGRAL

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The following integral is of frequent occurrence in Calculus:

If p, q, m, n are integers, and in addition m and n are >1, then

$$\int_{p\pi/2}^{q\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \cdot \int_{p\pi/2}^{q\pi/2} f(\theta)\, d\theta$$

where
$$f(\theta) = \begin{cases} \sin\theta\cos\theta, & \text{if } m \text{ and } n \text{ are both odd,} \\ \sin\theta, & \text{if } m \text{ is odd and } n \text{ even,} \\ \cos\theta, & \text{if } m \text{ is even and } n \text{ odd,} \\ 1, & \text{if } m \text{ and } n \text{ are both even,} \end{cases}$$

and in the numerical coefficient the number of factors in the denominator and the number of factors in the numerator are equal.

The proof is simple, and the form given has several advantages.

1. The proof is based on the integration by parts

$$\int \sin^m\theta \cos^n\theta\, d\theta = \frac{-\sin^{m-1}\theta\cos^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}\theta\cos^n\theta\, d\theta.$$

This equation implies that

$$\int_{p\pi/2}^{q\pi/2} \sin^m\theta\cos^n\theta\, d\theta = \frac{m-1}{m+n}\int_{p\pi/2}^{q\pi/2} \sin^{m-2}\theta\cos^n\theta\, d\theta,$$

and there is a corresponding form for reducing the exponent on $\cos\theta$.

2. It is easily remembered, for the form of statement keeps in evidence the principle that is being used.

In application one can write down the coefficient as the quotient of two products, followed by the integral for that case.

Thus

$$\int_{\pi/2}^{2\pi} \sin^7\theta\cos^5\theta\, d\theta = \frac{6.4.2.4.2}{12.10.8.6.4}\int_{\pi/2}^{2\pi} \sin\theta\cos\theta\, d\theta$$

$$= \frac{6.4.2.4.2}{12.10.8.6.4}\left[\frac{\sin^2\theta}{2}\right]_{\pi/2}^{2\pi}$$

$$= -\frac{1}{120}.$$

Another advantage of the form is that the interval of integration $p\pi/2$ to $q\pi/2$ is general, whereas in the forms given in many Calculus texts as Wallis's formulae the interval of integration is 0 to $\pi/2$. The form here given, however,

covers all three of Wallis's forms as special cases, and in addition it applies, without modification, to whatever interval of integration turns up in the application being made.

ON A CHAINOMATIC ANALYTICAL BALANCE

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In the type of chainomatic analytical balance shown in Figure 1, one end of a flexible gold chain is attached to one end of the cross beam and the other end is attached to a slider on a vertical bar which can be moved up and down the bar by means of a crank. The scale on the vertical bar is linear and consequently there is a vernier attachment.

On asking many chemists why the scale was linear, the most frequent response was "it's obvious." The purpose of this note is to determine the character of the vertical scale.

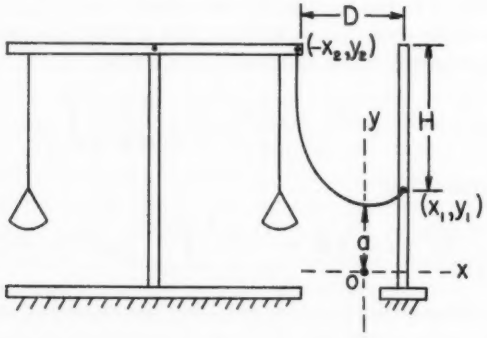

FIG. 1

Let the length and density of the chain be denoted by L and ρ, respectively Also, let D denote the distance that the vertical bar is from the end of the cross beam, and let H denote the distance that the moving end of the chain is below the cross beam. Then since the chain forms a catenary, the following equations must hold:

$$a \sinh \frac{x_2}{a} + a \sinh \frac{x_1}{a} = L, \tag{1}$$

$$a \cosh \frac{x_2}{a} - a \cosh \frac{x_1}{a} = H, \tag{2}$$

$$x_2 + x_1 = D, \tag{3}$$

$$W = \rho a \sinh \frac{x_2}{a}. \tag{4}$$

Here, the equation of the catenary is taken in the form $y=a\cosh x/a$, and the coordinates of the fixed end and the movable end of the chain are taken as $(-x_2, y_2)$, and (x_1, y_1), respectively. W represents the weight added to the reading of the balance due to the chain.

The problem now is to eliminate x_1, x_2, and a from the four equations. If equations (1) and (2) are rewritten in the form

$$2a\sinh\left[\frac{x_2+x_1}{2a}\right]\cosh\left[\frac{x_2-x_1}{2a}\right]=L, \tag{5}$$

$$2a\sinh\left[\frac{x_2+x_1}{2a}\right]\sinh\left[\frac{x_2-x_1}{2a}\right]=H, \tag{6}$$

it then follows that

$$\tanh\left[\frac{x_2-x_1}{2a}\right]=\frac{H}{L}, \tag{7}$$

and

$$2a\sinh\left[\frac{D}{2a}\right]=\sqrt{L^2-H^2}. \tag{8}$$

Consequently,

$$W=\rho a\sinh\left[\frac{D}{2a}+\tanh^{-1}\frac{H}{L}\right]=\frac{\rho L}{2}+\frac{\rho H}{2}\coth\left[\frac{D}{2a}\right]. \tag{9}$$

It does not follow from (9) that W is linear in H, since a is a function of H as given by (8). However, it will be shown that by suitably restricting the values of L, H, and D, W can be made to be approximately linear in H, the approximation being sufficiently good for the purpose of the balance (*i.e.*, to weigh to a tenth of a milligram).

To find $D/2a$ from equation (8) we plot

$$y=\sinh x, \quad \text{and} \quad y=x\frac{\sqrt{L^2-H^2}}{D}. \tag{10}$$

Here $x=D/2a$. If H is restricted to the range $0\leqq H\leqq L/2$ and $D\ll L$, then it follows from the plot of (10) that $a\ll D$ (the solution $a=\infty$ does not apply here). Now

$$\coth\left[\frac{D}{2a}\right]=1+\frac{2e^{-D/a}}{1-e^{-D/a}}, \tag{11}$$

and since $1\gg 2e^{-D/a}[1-e^{-D/a}]^{-1}$, it follows that W is approximately linear in H.

If for example $L=10D$, then $D/2a>4.3$ (for $H=L/2$), and

$$e^{-D/a}[1 - e^{-D/a}]^{-1} < \frac{1}{5000}.$$

Since the chain is designed to weigh from 0.0000 to 0.1000 grams (by adjusting ρL), the error in assuming a linear scale (which allows the use of a vernier) is less than 0.00004 grams which is negligible for the purpose of the balance.

Remark. This example tends to indicate that chemists as well as mathematicians are prone to over-use the word "obvious".

A PROOF OF LEGENDRE'S DUPLICATION FORMULA

S. K. Lakshmana Rao, Indian Institute of Science

The following proof of the formula

$$\Gamma(2s) = (2^{2s-1}/\sqrt{\pi})\Gamma(s)\Gamma(s + \tfrac{1}{2})$$

by the use of the Mellin Transform is interesting.

The Mellin Transform of a function $f(x)$ of one variable, defined for positive values of the argument, is by definition $Mf(x) = \int_0^\infty f(x)x^{s-1}dx$. We have the Faltung theorem,*

$$Mf_1(x) \cdot Mf_2(x) = Mg(x) \tag{1}$$

where

$$g(x) = \int_0^\infty f_1\left(\frac{x}{u}\right) f_2(u)\,\frac{du}{u}.$$

Now take

$$f_1(x) = e^{-x}, \qquad f_2(x) = e^{-x}x^{1/2}.$$

Then

$$Mf_1(x) = \Gamma(s), \qquad Mf_2(x) = \Gamma(s + \tfrac{1}{2}) \tag{2}$$

and

$$g(x) = \int_0^\infty e^{-x/u}e^{-u}u^{1/2}\,\frac{du}{u} = \int_0^\infty e^{-(x/u+u)}\,\frac{du}{u^{1/2}},$$

---

* Sneddon, I. N., Fourier Transforms, McGraw-Hill, 1951, p. 43.

Readers interested in Laplace, Fourier, Hankel, Mellin, and other transforms are also referred to the series of books currently being prepared by the Staff of the Bateman Manuscript Project at the California Institute of Technology and published by McGraw-Hill; in particular, Tables of Integral Transforms, 1954.

the value of which is easily seen to be $\sqrt{\pi}\, e^{-2\sqrt{x}}$. For

$$g(x) = 2\int_0^\infty e^{-(x/t^2+t^2)}dt = 2\int_0^\infty e^{-(x/t^2+t^2)}\frac{x^{1/2}}{t^2}\,dt,$$

where the last integral is derived from the previous one by changing t to $x^{1/2}/t$. Therefore

$$g(x) = \int_0^\infty e^{-(x/t^2+t^2)}\left(1+\frac{x^{1/2}}{t^2}\right)dt.$$

Now put $t-x^{1/2}/t=u$. Then

$$g(x) = \int_{-\infty}^\infty e^{-(u^2+2x^{1/2})}du = e^{-2x^{1/2}}\int_{-\infty}^\infty e^{-u^2}du = \sqrt{\pi}e^{-2x^{1/2}},$$

$$Mg(x) = \sqrt{\pi}\int_0^\infty e^{-2x^{1/2}}x^{s-1}dx.$$

Finally set $2x^{1/2}=y$, $dx=y\,dy/2$, and obtain

$$Mg(x) = \sqrt{\pi}\int_0^\infty e^{-y}\left(\frac{y}{2}\right)^{2s-1}dy = \frac{\sqrt{\pi}\Gamma(2s)}{2^{2s-1}}.$$

Therefore from (1) and (2)

$$\Gamma(2s) = (2^{2s-1}/\sqrt{\pi})\Gamma(s)\Gamma(s+\tfrac{1}{2}).$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1151. *Proposed by C. S. Ogilvy, Hamilton College*

Resolve the following paradox. The area between the curve $y=1/x$ and the x-axis, to the right of the line $x=1$, is infinite. Yet the volume generated by rotating this area about the x-axis is π. Thus it would require an infinite quantity of paint to cover the area; yet the volume, which completely contains and surrounds the area, could be filled with π cubic units of paint!

E 1152. *Proposed by J. J. Bowers, Student, Wesleyan University*

If, in the postulates of a field, the distributivity of multiplication over addition is replaced by distributivity of addition over multiplication, is the resulting set of postulates consistent?

E 1153. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

For any angle θ, show that arbitrarily small constructible angles ϕ exist such that $(\theta-\phi)$ can be trisected.

E 1154. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The distance from the midpoint of side AB of a regular convex heptagon $ABCDEFG$, inscribed in a circle, to the midpoint of the radius perpendicular to BC and cutting this side, is equal to half the side of a square inscribed in the circle.

E 1155. *Proposed by Donald Bratton, Cowles Commission, Chicago*

There is a group of people who habitually receive parcels. When one of these persons x receives a parcel addressed to y, he relays it to $f(x, y)$. The function f is such that a parcel will eventually reach the person to whom it is addressed, no matter who first receives it.

One sees that a person x for which $f(x, y)$ takes on only a single value as y runs over all the values $\neq x$, is not very important. So let us call the *valence* of x the number of values assumed by $f(x, y)$ for $y\neq x$, less one. The unimportant people are thus those with zero valence. Denote by V the sum of all the valences in the group.

It is easy to show that, if $V=0$, a parcel can be addressed and started out in such a way that each person in the group will handle it. On the other hand, can you prove that, if $V\neq 0$, $V+1$ parcels can be addressed and started out so that each person in the group will handle at least *two* of these parcels?

SOLUTIONS

A Property of Any Collection of Integers

E 1121 [1954, 423]. *Proposed by W. D. Serbyn, Carnegie Institute of Technology*

Let E be any collection of n integers, not necessarily distinct. Show that there exists a non-empty subcollection $F\subseteq E$ such that the sum of the integers contained in F is divisible by n.

Solution by L. R. Ford, Illinois Institute of Technology. Let $a_1, \cdots, a_n$ be the integers of E. Of the $n+1$ integers $0, a_1, a_1+a_2, a_1+a_2+a_3, \cdots, a_1+\cdots+a_n$ two at least are congruent modulo n. Their difference, which is the sum of the integers in a subset of E, is divisible by n.

Also solved by J. T. Ahlin, J. L. Botsford, W. E. Briggs, A. L. Epstein, S. H. Gould, D. S. Greenstein, B. A. Hausmann, A. R. Hyde, P. B. Johnson,

W. S. Loud, D. C. B. Marsh, Leo Moser, D. B. Mumford, J. H. Oppenheim, F. D. Parker, J. V. Pennington, R. R. Phelps and J. L. Selfridge (jointly), L. L. Scott, R. P. Tapscott, and the proposer. Late solution by T. F. Mulcrone.

Moser pointed out that this problem is a very special case of his problem 4300. See the remark, following the solution, by J. B. Kelly [1950, 47]. Problem 4300 states that every set of n (not necessarily distinct) elements of a finite group of order n contains a subset of elements whose product is the unit element.

Stacking Cards

E 1122 [1954, 423]. *Proposed by P. B. Johnson, Occidental College and Haverford College*

Perfectly rigid playing cards are piled on the edge of a table with the pile slanting up away from the table. How far from the edge of the table can the pile be made to extend without falling to the floor?

I. *Solution by Michael Goldberg, Washington, D. C.* Let a_i represent the distance the ith card from the top extends beyond the edge of the table. The limiting position which a succession of cards may take is one in which the center of gravity of any top set is directly above the edge of the card next below. Hence

$$(a_1 - 1/2) + (a_2 - 1/2) + \cdots + (a_k - 1/2) = ka_{k+1}.$$

From this equation it follows that

$$a_1 = a_2 + 1/2,\ a_2 = a_3 + 1/4,\ \cdots,\ a_k = a_{k+1} + 1/2k.$$

Therefore, for n cards, the distance a_1 is given by

$$a_1 = (1/2)(1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n).$$

As n increases without limit, a_1 increases without limit, because the harmonic series is divergent.

II. *Solution by Albert Wilansky, Lehigh University.* The cards can extend any distance. We see this thus: Place one card on top of a vertical stack so that its edge extends out over the top; the vertical stack is assumed to have so many cards in it that the center of gravity of the system is a certain small distance ϵ, measured horizontally, from the center of the vertical stack. Now place the whole system on top of a vertical stack with the bottom of this system projecting over the edge of this lowest vertical stack; the lowest stack is assumed to have so many cards in it that the center of gravity of the whole (new) system is horizontally within ϵ of the center of the bottom card. This may be kept up indefinitely and eventually the edge of the highest card will be any desired distance from the edge of the lowest, and the center of gravity of the system will be horizontally within ϵ of the center of the lowest card.

If the cards are of zero thickness the highest card will be at the same level as the lowest. The system will then look like a long thin board miraculously

balanced near one end, the secret being that the board, like a child's toy, is denser at one end than the other.

Also solved by S. H. Eisman, A. R. Hyde, P. G. Kirmser, L. V. Mead, Leo Moser, George Mott, D. B. Mumford, C. S. Ogilvy, C. R. Perisho, Azriel Rosenfeld, and the proposer. Late solution by M. S. Klamkin.

Moser pointed out that this problem is essentially the same as problem 52 of the April 1954 *Pi Mu Epsilon Journal.*

Several solvers noted that, by the method of stacking of solution I, the top card of a deck of 52 cards will project approximately 2.268 card lengths beyond the edge of the table.

Polygonal Path Covering a Square Lattice

E 1123 [1954, 423]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Given a square $N \times N$ point lattice, show that it is possible to draw a polygonal path passing through all the N^2 lattice points and consisting of $2N-2$ segments. Can it be done with less than $2N-2$ segments?

Solution by the Proposer. Assume that it can be done for $N=K$ in such a fashion that we end up in position E as shown in Figure 1. Then, as shown in

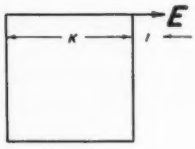

FIG. 1

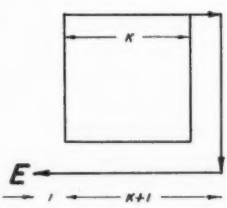

FIG. 2

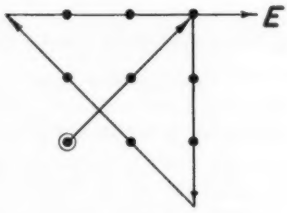

FIG. 3

Figure 2, by drawing two more lines it can be done in the same fashion for $N=K+1$. Figure 3 shows that it can be done in this fashion for $N=3$. Thus, by induction, it can be done for all $N \geqq 3$.

Also solved by G. C. Bush, Michael Goldberg, Dudley Herschbach, and D. C. B. Marsh.

No one successfully answered the question at the end of the problem.

Iterated Exponents

E 1124 [1954, 423]. *Proposed by L. J. Lander and J. L. Selfridge, U. C. L. A.*

Let $(a;0)=1$ and $(a;n+1)=a^{(a;n)}$. Find all solutions of $(a;m)=(b;n)$ in integers a, b, m, n all greater than 1.

Solution by the Proposers. Clearly $a=c^r$ and $b=c^s$. We can choose c so that $(r,s)=1$. The exponents of c in the given equation are $r(a;m-1)=s(b;n-1)$.

We may assume $r \leqq s$, and it follows that $s = rc^d$ so that $r = 1$, $c = a$, and $b = a^{s^d}$. The exponents of a in the equation $(a;\ m-1) = a^d(b;\ n-1)$ are $(a;\ m-2) = d + a^d(b;\ n-2)$. Now assume that k is the largest integer such that $(a;\ k) \mid d$. Clearly a^d and hence $(a;\ m-2)$ are divisible by $(a;\ k+1)$; *i.e.*, there is no such integer k. This can occur if $a = 1$ or $d = 0$. Since $a > 1$, we have $a = b$ and $m = n$.

Stirling Numbers

E 1125 [1954, 423]. *Proposed by Walter James, University of Minnesota*

Determine the coefficients B_j^p for the following sum:

$$\sum_{s=1}^{n} s^p = \frac{1}{(n-1)!} \sum_{j=1}^{p} \frac{B_j^p (n+j)!}{j+1}.$$

Solution by A. R. Hyde, West Hartford, Conn. If S_n represents the sum of n terms, then

$$(S_n - S_{n-1})/n = n^{p-1} = B_1^p + (n+1)B_2^p + (n+1)(n+2)B_3^p + \cdots + (n+1)\cdots(n+p-1)B_p^p.$$

By setting n equal successively to $-1, -2, \cdots, -(p-1)$, the constants may be determined so that this is an identity. Then

$$B_j^p = \frac{1}{(j-1)!} \sum_{k=1}^{j} (-1)^{p+k}\, {}_{j-1}C_{k-1} k^{p-1} = \sum_{k=1}^{j} (-1)^{p+k} \frac{k^{p-1}}{(j-k)!(k-1)!}.$$

Values of B_j^p for $j, p \leqq 9$ are as follows:

p \ j	1	2	3	4	5	6	7	8	9
1	1	·	·	·	·	·	·	·	·
2	−1	1	·	·	·	·	·	·	·
3	1	−3	1	·	·	·	·	·	·
4	−1	7	−6	1	·	·	·	·	·
5	1	−15	25	−10	1	·	·	·	·
6	−1	31	−90	65	−15	1	·	·	·
7	1	−63	301	−350	140	−21	1	·	·
8	−1	127	−966	1701	−1050	266	−28	1	·
9	1	−255	3025	−7770	6951	−2646	462	−36	1

Also solved by H. W. Gould, A. S. Grant, D. C. B. Marsh, Chih-yi Wang, and the proposer. Late solution by M. S. Klamkin.

Wang showed that if p is odd $B_j^p = (-1)^{j-1} S_p^j$ and if p is even $B_j^p = (-1)^j S_p^j$, where S_p^j is a Stirling number of the second kind. The calculation of these

Stirling numbers is indicated in Charles Jordan, *Calculus of Finite Differences*, and are there tabulated for $j, p \leqq 12$.

Hyde's table is easily calculated and extended if we note (as did the Proposer) that $B_j^p = (-1)^{p+j} A_j^p$, where $A_j^p = jA_j^{p-1} + A_{j-1}^{p-1}$ and $A_1^1 = 1$. Thus, after putting 1's in the first column and along the diagonal, the jth element of any row will numerically equal j(number immediately above)+(number immediately above to the left). For example, in Hyde's table, $90 = 3(25) + 15$, $1050 = 5(140) + 350$, *etc.* The signs are then taken care of by making the terms of the even diagonals negative.

In addition to solving the problem, Gould compared several known alternative methods for expanding $\sum_{i=1}^{n} s^p$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4623. *Proposed by L. L. Pennisi, University of Illinois, Navy Pier, Chicago*

Prove that

$$\lim_{x \to \infty} \frac{\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{3 \cdot 5 \cdots (2n+1)}}{\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2 \cdot 4 \cdots 2n}} = \sqrt{\frac{\pi}{2}}.$$

4624. *Proposed by H. S. Shapiro, New York University*

Let J_p denote the field of integers modulo p, where p is a prime, and $J_p[x]$ the ring of polynomials with coefficients in J_p. Then the number of monic irreducible polynomials of degree n in $J_p[x]$ is equal to

$$\frac{1}{n} \sum_{d \mid n} \mu(d) p^{n/d}$$

where μ is the Möbius function.

4625. *Proposed by K.-F. Moppert, Basle, Switzerland*

For $a>0$, $b>-1$, prove

$$\frac{1}{a}-\binom{b}{1}\frac{1}{1+a}+\binom{b}{2}\frac{1}{2+a}-+\cdots=\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(1+a+b)}.$$

4626. *Proposed by W. A. Michael and D. A. Page, University of Illinois, Urbana*

Let A be a subset of the plane. Join each pair of distinct points of A by a line. Let A_1 consist of the points of A together with the points of intersection of distinct lines. Starting with A_1 the same construction yields a set A_2. In this way an increasing sequence of sets $A_1, A_2, \cdots$, is obtained. Let $\Omega=\cup A_n$. If A is denumerable then so is Ω, and hence Ω is not the entire plane; however, under what conditions will Ω be dense in the plane?

4627. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

In Chrystal, *Textbook of Algebra*, vol. 2, p. 225, there is the following theorem on the representation of an irrational number:

The number represented by the series

$$\sum_{n=1}^{\infty}\frac{p_n}{r_1r_2\cdots r_n}$$

is irrational provided that

(1) r_n and p_n are integers such that $0<p_n<r_n$,

(2) $r_{n+1}>r_n>1$,

(3) the sequence $\{r_1r_2\cdots r_n\}$ includes all powers of the primes.

(A) Construct a counter-example, *i.e.*, a number of the stated form which is rational even though conditions (1), (2) and (3) are satisfied.

(B) Complete the list of conditions (1), (2) and (3) so that the theorem is indeed valid.

SOLUTIONS

Two Triangles Inscribed in a Conic

4562 [1953, 632]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, let P be a point having normal coördinates (x, y, z) and consider the points A', B', C' with normal coördinates $(-x/2, y, z)$, $(x, -y/2, z)$, $(x, y, -z/2)$. (1) The points A, B, C, A', B', C' lie on one conic S, and there is a conic with respect to which the triangles ABC and $A'B'C'$ are self-polar. (2) If AP, BP, CP cut BC, CA, AB in A_1, B_1, C_1 and if $A'P$, $B'P$, $C'P$ cut $B'C'$, $C'A'$, $A'B'$ in A_1', B_1', C_1', the triangles ABC, $A'B'C'$ are circumscribed about a conic Σ, the points of tangency being A_1, B_1, C_1 and A_1', B_1', C_1'. (3) The conics S and Σ

have double contact along the common polar of P with respect to these conics.

Solution by Roscoe Woods, State University of Iowa. Let (X, Y, Z) denote current coördinates. The equation of any conic, K, circumscribing the reference triangle is of the form $l/X+m/Y+n/Z=0$, where l, m, n are constants. Also the equation of any conic, K', inscribed in the reference triangle is of the form $\sqrt{LX}+\sqrt{MY}+\sqrt{NZ}=0$, where L, M, N are constants and where the ambiguous signs before the radicals are omitted for convenience. Consider now the conic K'': $(L/l)X^2+(M/m)Y^2+(N/n)Z^2=0$ for which the reference triangle is self-polar. It is easily verified that the conics K and K' are the polar reciprocals of one another with respect to the conic K''. (See C. Smith, *Conic Sections*, 1914, p. 384.)

A short calculation shows that the conic S of the problem has the equation $x/X+y/Y+z/Z=0$ and the conic for which the triangles ABC and $A'B'C'$ are self-polar has the equation $X^2/x^2+Y^2/y^2+Z^2/z^2=0$. It is also readily seen that the conic Σ referred to in the problem has the equation $\sqrt{X/x}+\sqrt{Y/y}+\sqrt{Z/z}=0$.

Part (3) is answered through use of the following theorem: *If two conics have double contact, there is a linear combination of their equations which consists of the square of the equation of their chord of contact.* Now

$$\Sigma + 4S = (X/x + Y/y + Z/z)^2 = 0,$$

where $X/x+Y/y+Z/z=0$ is the common polar of $P(x, y, z)$ with respect to the conics Σ and S. It should be pointed out that this common polar is the trilinear polar of P with respect to the triangle ABC, being the axis of perspective of the triangles ABC and $A_1B_1C_1$.

Also solved by the Proposer.

Editorial Note: The results of this problem may be obtained easily from known theorems as given in Coxeter, *The Real Projective Plane*, p. 80:

6:71. *If two triangles are self-polar for a given polarity, their six vertices lie on a conic and their six sides touch another conic.*

6:72. *Any two triangles inscribed in a conic are self-polar for some polarity.*

A Sequence Almost Contained in a Set of Sequences

4563 [1953, 716]. *Proposed by Albert Edrei, Syracuse University*

Let

$$\alpha_1, \alpha_2, \alpha_3 \cdots \tag{1}$$

be an infinite, strictly increasing sequence of positive integers, and let $B=\{b\}$ be a set of such sequences. A sequence of the form (1) is said to be *almost contained* in B if, to every b, there corresponds an integer $k=k(b)$ such that α_k, α_{k+1}, α_{k+2}, $\cdots$ is a subsequence of b.

The following theorem is easy to prove: *Let B be denumerable and such that*

the intersection of any finite number of b's is an infinite sequence. Then there exists an infinite sequence almost contained in B.

The problem is to prove that this theorem is false if the word denumerable is omitted.

Solution by Paul Erdös, University of Notre Dame. If one takes B to be the class of all sequences of integers of density 1, one obtains a solution. Clearly the intersection of any finite number of sequences of density 1 again has density 1 and is thus infinite. On the other hand clearly no infinite sequence is almost contained in B since if $a_1<a_2<\cdots$ is an infinite sequence, it contains an infinite subsequence $b_1<b_2<\cdots$ of density 0, and the complement of the b's has density 1 and does not almost contain the a's.

Also solved by B. J. Ball, G. E. Bredon, Hewitt Kenyon, O. Mourmaki, J. D. Reid, and the Proposer.

Another Summation

4564 [1953, 716]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Prove

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}=\frac{\pi^4}{72}, \quad \text{where} \quad S_n=\sum_{r=1}^{n}\frac{1}{r}.$$

I. *Solution by J. V. Whittaker, University of California, Los Angeles.* We have, after rearrangement of the terms of the series,

$$S=\sum_{m=1}^{\infty}\sum_{n=1}^{m}\frac{1}{nm^3}=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n(m+n)^3}.$$

We first notice that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2(m+n)^2}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^2n^2}-\sum_{m=1}^{\infty}\sum_{n=1}^{m}\frac{1}{m^2n^2}=\frac{\pi^4}{120}.$$

Then

$$\sum_{n=1}^{\infty}\frac{1}{n(m+n)^3}=\sum_{n=1}^{\infty}\left\{\frac{1}{m^3n}-\frac{1}{m^3(m+n)}-\frac{1}{m^2(m+n)^2}-\frac{1}{m(m+n)^3}\right\}$$

$$=\sum_{n=1}^{m}\frac{1}{m^3n}-\sum_{n=1}^{\infty}\left\{\frac{1}{m^2(m+n)^2}+\frac{1}{m(m+n)^3}\right\}.$$

Finally, summing over m from 1 to ∞, we find that

$$\left(S-\frac{\pi^4}{90}\right)=S-\frac{\pi^4}{120}-\left(S-\frac{\pi^4}{90}\right),$$

or $S=\pi^4/72$.

II. *Solution by the Proposer.* This result is a special case of a theorem due to G. T. Williams (this MONTHLY, 1953, p. 25) which may be put in the form

$$2\sum_{n=1}^{\infty}\frac{S_n}{n^p}=(p+2)\zeta(p+1)-\sum_{j=1}^{p-2}\zeta(j+1)\zeta(p-j).$$

For $p=3$, we have

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}=\frac{1}{2}\left\{5\zeta(4)-\zeta^2(2)\right\}=\frac{\pi^4}{72}.$$

It may be obtained as easily from a result due to D. H. Browne in connection with Problem no. 4431 [1952, 472].

Also solved by W. E. Briggs and S. Chowla and P. C. Rosenbloom, Leonard Carlitz, A. E. Livingston, O. E. Stanaitis, Ernst Trost, and Chih-yi Wang.

Unit Element of a Valuation Ring

4565 [1953, 716]. *Proposed by Donald Bratton, University of Chicago*

A ring A is called a valuation ring when it is commutative and, for each couple of elements (x, y) of A, either x divides y or y divides x. Show that each valuation ring has a unit element.

Solution by D. W. Sasser, Student, Yale University. For any non-zero element x, there is a y such that $xy=x$. Every non-zero element z of A is either a multiple or a divisor of x. If $cx=z$, then $zy=cxy=cx=z\neq 0$. If $x=dz$, then $dzy=xy=x\neq 0$, whence $zy\neq 0$. Thus y is not a divisor of zero.

Now there is an a such that $ya=y$. Then for each z in A, $y(az-z)=0$, whence $az=z$. Thus a is a unit element.

Also solved by Rafael Chacon and Simon Hellerstein, I. S. Cohen and Oscar Goldman, C. H. Denbow, W. E. Deskins, M. P. Epstein, D. C. B. Marsh, R. S. Pierce, J. D. Reid, Alex Rosenberg, D. W. Sasser, W. R. Scott and Elbert Walker, David Shale, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Differential equations with applications. By Herman Betz, P. B. Burcham, and G. M. Ewing. New York, Harper, 1954. 10+310 pp. $4.50.

To quote from the preface "this is a text for an introductory course in differential equations, emphasizing particularly applications to the physical and

biological sciences and to engineering." Indeed this book has more than the usual complement of examples from the sciences, and in writing a text for the student oriented toward the applications, the authors have succeeded admirably. From the mathematical standpoint stress is placed on a variety of methods which work. For example, the formalities of the Laplace transform method are included. Questions of existence, uniqueness, and convergence receive little attention. On the other hand, the geometry of solutions is treated in detail in some cases; an example being a discussion of a limit cycle of a nonlinear equation of the van der Pol type.

An introductory chapter, which shows how differential equations arise, is followed by one on equations of the first order and degree, including Ricatti's equation. The third chapter is devoted to a large number of applications of first order equations. For example, the law of mass action is discussed in detail, complete with illustrative chemical formulas. The mutual relationship between sharks and soles in the Adriatic, which Volterra translated into a pair of equations of the first order, is given a geometric treatment. Methods of solving some implicit equations are presented in Chapter 4. Linear equations are next studied, with the "operator method" used for equations with constant coefficients. (The inverse operator introduced on page 94 does not seem to be uniquely defined.) Mechanical and electrical applications of linear equations with constant coefficients are treated in some detail. Formal properties of the Laplace transform are given and illustrated by electric circuit and beam problems. Special types of higher order equations are discussed in Chapter 8. Series solutions are introduced in Chapter 9, along with an optional section on regular singular points of second order linear equations. Chapter 10, which is devoted to graphical and numerical methods, includes the example of a limit cycle, and Runge's method. The last two chapters are on partial differential equations, Fourier series being used for heat flow and vibration problems. An appendix on hyperbolic functions is given.

Topics omitted which are of interest include the geometry of solutions of a pair of first order linear equations with constant coefficients, and boundary value problems involving Laplace's equation in two dimensions. However, the book contains more than enough for a semester's course.

There is a large number of exercises, and references for supplementary reading are given at the end of many of the chapters. It is unfortunate that references were not given for the last chapter.

The printing is excellent; no typographical errors were noticed. There might be some confusion between the notation $(\partial E/\partial S)_v$ on page 245 and $(\partial\theta/\partial x)_x$ on page 259.

E. A. CODDINGTON
University of California at Los Angeles.

Elementary Theory of Numbers. By Harriet Griffin. New York, McGraw-Hill Book Company, 1954. ix+203 pages. $5.00.

This book stresses the elementary topics in the arithmetic theory of numbers and is written primarily for the undergraduate or immature graduate student. The author states in the preface that the presentation has been classroom tested and has proved successful. Most of the theorems are proved in detail. There are, however, some proofs which must be furnished by the careful reader. Most of the theorems are illustrated by examples, a feature that will appeal especially to the undergraduate. The problems are numerous and the instructor will find among them challenges for students of all abilities.

The chapter headings are as follows: 1. The Fundamental Laws; 2. The Linear Diophantine Equation; 3. Properties of Integers; 4. Properties of Congruences; 5. The Solution of Congruences; 6. The Theorems of Fermat and Wilson and the Möbius Function; 7. On Belonging to an Exponent; 8. Indices; 9. Quadratic Residues; 10. Some Famous Problems; 11. Polynomials; 12. Partitions. One unusual aspect of this ordering is that the Euclidean algorithm appears in Chapter 3, after the chapter on linear diophantine equations. The method of solution given in Chapter 2 is essentially equivalent to the method which uses the Euclidean algorithm. Chapters 5 to 9 are perhaps more exhaustive in the treatment of the topics involved than most elementary texts on number theory. The Waring problem, the equation $x^2+y^2=z^2$, and Fermat's last theorem are among the problems discussed in the chapter on famous problems. The chapter on polynomials begins with sets of postulates for integral domains and fields and continues with the development of some of the elementary properties of the domain of polynomials over a field. The additive theory of numbers is introduced very briefly in the final chapter.

Throughout the book brief historical highlights concerning the topic being discussed are cited. Another feature is the frequent footnote reference to periodical literature, some very recent, and most of which is within the limitations of the average student. It is the opinion of this reviewer that the book presents in an interesting manner adequate background material for students wishing to go on to advanced work.

No serious typographical errors were noted but a consistent omission of punctuation marks at the ends of mathematical expressions was noted.

V. J. VARINEAU
University of Wyoming

An Introduction to the Calculus of Finite Differences. By C. H. Richardson. New York, D. Van Nostrand Co., Inc., 1954. 6+142 pages. $5.00.

This book is precisely what its title indicates. It is an introduction to finite differences. Its contents are based on those of a semester course given by the author as an elective for juniors and seniors. It was designed primarily for students in actuarial and statistical theory at this level but it should also be of value to other undergraduates interested in applied mathematics.

The material of the book is divided into six chapters and two short appendices. The first chapter is introductory in nature and treats the basic definitions, difference formulas, and symbolic operators. The initial definitions are given for a general $\Delta x = h$ but after page 2 the discussion is limited to $\Delta x = 1$.

Chapter 2 discusses finite integration, summation of series, Stirling's numbers, and the differences of zero. Chapter 4, consisting of 24 pages, is devoted to interpolation and approximate integration. A brief introduction to the interpolation formulas of Newton, Gauss, Stirling, Bessel, and Lagrange is followed by some simple quadrature formulas and the Euler-Maclaurin sum formula.

A short Chapter 5 is devoted to Beta and Gamma functions while Chapter 6 gives an introduction to the treatment of difference equations. Brief appendices appear on mathematical induction and hyperbolic functions. (The only prerequisite assumed is an elementary course in infinitesimal calculus.)

The book is concisely yet clearly written in a style which is appropriate to the undergraduate. Occasional references are given for the benefit of the student who may wish to study some topic in detail. The illustrations are adequate and the exercises more than ample for the purpose. The author and the publisher should be commended for producing this book which provides a concise elementary text in finite differences.

P. S. DWYER
University of Michigan

NEW BOOKS RECEIVED

Tables of Integral Transforms, Vol. II. By the Bateman Project Staff. Editor, A. Erdelyi. New York, McGraw-Hill Book Company, 1954. xvi+451 pages. $8.00.

Der Mathematische Unterricht fur die sechzehn-bis einundzwanzig Jährige Jugend in der Bundesrepublik Deutschland. By Heinrich Behnke. Göttingen, Vandenhoeck and Ruprecht, 1954. 332 pages. 20 DM.

Schaum's Outline of Theory and Problems of Plane and Spherical Trigonometry. By Frank Ayres, Jr. New York, Schaum Publishing Company, 1954. 207 pages. $1.85.

Understanding Numbers: Their History and Use (A Telecourse Syllabus). By P. S. Jones. Ann Arbor, Braun-Brumfield; available at Ulrich's Bookstore, 547–549 E. University Avenue, Ann Arbor, Michigan. 1954. 50 pages. $1.00.

An Experimental Investigation of the Mechanics of Plastic Deformation of Metals. By E. G. Thomsen, C. T. Yang, and J. B. Vierbower. Berkeley 4, California, University of California Press, 1954. 55 pages. $.75.

Graphical Method of Statistical Inference. By M. Masuyama. Tokyo, Maruzen Company, Ltd., 1954. $2.50.

Table of Sine and Cosine Integrals for Arguments from 10 to 100. By U. S.

Department of Commerce, National Bureau of Standards (Applied Mathematics Series—32). Washington, D. C., Government Printing Office, 1954. 186 pages. $2.25.

Table of the Gamma Function for Complex Arguments. By U. S. Department of Commerce, National Bureau of Standards. (Applied Mathematics Series—34). Washington, D. C. Government Printing Office, 1954. 105 pages. $2.00.

Mathematics in Western Culture. By Morris Kline. New York, Oxford University Press, 1954. xii+484 pages. $6.00.

Technical Mathematics. By H. S. Rice and R. M. Knight. New York, McGraw-Hill Book Company, 1954. xiv+748 pages. $6.50.

Elements of Algebra. By Howard Levi. New York, Chelsea Publishing Company, 1954. 160 pages. $3.25.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER INSTITUTES FOR HIGH SCHOOL AND COLLEGE TEACHERS

The National Science Foundation will sponsor three institutes for teachers of mathematics in the summer of 1955. Preliminary announcements of these appear below. In each case housing will be provided in university dormitories and a limited number of stipends will be available for participants.

University of Wisconsin. This institute for both high school and college teachers will be held during the period June 27 to July 23 under the direction of Professor C. C. MacDuffee, to whom inquiries should be directed. It will stress relations between the college mathematics teacher and the high school or preparatory school teacher. Lectures will be given for each of these groups separately and jointly. The problem of the education of teachers of mathematics will also receive attention.

Oklahoma A. and M. College. This institute for college teachers will be held during the period June 13 to July 22 under the direction of Professor L. Wayne Johnson, to whom inquiries should be directed. The principal lecturers will be Professor Arthur Rosenthal of Purdue University, whose topic will be "An Introduction to the Theory of Measure and Integration"; and Professor H. S. M. Coxeter of the University of Toronto, whose topic will be "Contributions of Geometry to the Main Stream of Mathematics." Part-time lecturers will be Professor A. W. Tucker of Princeton University, who will speak on "An Introduction to the Theory of Games"; Professor Saunders MacLane of the University

of Chicago, whose topic is "The Undergraduate Program in Mathematics in the University of Chicago"; and Professor G. B. Price of the University of Kansas, who will discuss "A Universal Course in Mathematics." The sessions of the Institute will be in air-conditioned buildings.

Stanford University. This institute for college teachers will be held during the period June 27 to August 19 under the direction of Professor Harold M. Bacon, P. O. Box 1144, Stanford, California, to whom inquiries should be directed. One of the two chief lecturers will be Professor George Pólya of Stanford University, who will speak on "Calculus—Its History, Application, and Teaching." The other main speaker will be Professor I. J. Schoenberg of the University of Pennsylvania, who will discuss a topic in the field of applied mathematics. There will be shorter series of lectures on "Numerical Analysis and Computing Machines" by Professor D. H. Lehmer and "The Problem of the Undergraduate Curriculum in Mathematics" by Professor C. B. Allendoerfer.

GRADUATE INSTITUTE FOR MATHEMATICS AND MECHANICS

Indiana University has established the Graduate Institute for Mathematics and Mechanics. It includes the members of the former Graduate Institute for Applied Mathematics and is an enlarged and strengthened body whose efforts are devoted to research and advanced graduate teaching in pure and applied mathematics. Students are required to have had at least one year's previous graduate study. The members of the regular staff are: Professors T. Y. Thomas (Director), V. Hlavatý, E. Hopf, C. Truesdell, J. W. T. Youngs; Associate Professors D. Gilbarg, W. Gustin. With the assistance of an international board of thirty specialists outside Indiana University, the Institute publishes the *Journal of Rational Mechanics and Analysis*, an international periodical for the fields of its title; it appears in bi-monthly issues making annual volumes of about 800 pages.

HONORARY MATHEMATICS FRATERNITY

Chapters of Pi Mu Epsilon, national honorary mathematics fraternity, may be chartered only in colleges and universities requiring at least eight semester hours beyond calculus for a mathematics major and having an average of at least five majors per year. The institution must have on its staff at least one person who has a Ph.D. degree *in mathematics* and must have had an active mathematics club for at least one year. Faculty members of institutions meeting these requirements may write to Professor R. V. Andree, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma, for further information.

SYMPOSIUM ON MATHEMATICS

The Second Symposium on "Some Mathematical Problems being Studied in Latin America," organized by the Unesco Science Cooperation Office for Latin America, was held at Mendoza, Argentina, during July 21 to 25, 1954, under the sponsorship of the University of Cuyo.

The following papers were read at the meetings:

1. *Some questions on affine differential geometry of surfaces*, by L. A. Santaló (Argentina).

2. *Some questions concerning topological vector spaces*, by L. Nachbin (Brazil).

3. *Multivalued analytic functions of Fourier transforms*, by A. Calderón (Argentina), in collaboration with R. Arens.

4. *On some divergent integrals of quantum electrodynamics*, by A. G. Domínguez (Argentina).

5. *Hilbert transforms of distributions*, by J. Horvath (Colombia).

6. *The problem of moments and Hermitian operators*, by M. Cotlar (Argentina).

7. *Absolute form of the transformation of the equations of dynamics in a curved space of n dimensions*, by G. García (Peru).

8. *Hydrodynamics: Recent advances in the theory of free boundaries*, by E. H. Zarantonello (Argentina).

9. *On some analytic applications of the expansions in series of Legendre polynomials*, by M. O. González (Cuba).

10. *Arithmetic of filters and topological spaces*, by A. Monteiro (Argentina).

11. *The theory of topological tensor products*, by A. Grothendieck (France).

12. *Algebraic operations in topology and some applications to geometric problems*, by J. Adem (Mexico).

13. *On some generalizations of the theory of functions of a complex variable*, by E. Lammel (Argentina).

14. *The functional equations of the theory of magnitudes*, by P. Pi Calleja (Argentina).

15. *A new definition of random function and its ergodic theorem*, by G. Dedebant (Argentina).

16. *Reductive subgroups of algebraic Lie groups*, by G. Mostow (U.S.A.).

17. *Problems about the definition of logic truth in semantic and syntactic systems*, by G. Klimovsky (Argentina).

18. *Singular integrals*, by A. Calderón (Argentina).

PERSONAL ITEMS

Columbia University announces the following: Assistant Professor I. M. Singer of the University of California at Los Angeles has been appointed Visiting Assistant Professor; Assistant Professor Harish-Chandra has been promoted to an associate professorship and was invited to attend a special colloquium in celebration of the centennial birth of Henri Poincaré at the University of Paris, France, October 17–31, 1954.

Harvard University reports: Visiting Assistant Professor J. T. Tate, Jr. of Columbia University has been appointed to an assistant professorship; Dr.

B. M. Dwork, formerly a student at Columbia University, has been appointed Benjamin Peirce Instructor.

At Illinois Institute of Technology: Associate Professor Haim Reingold has been named Chairman of the Department of Mathematics; Dr. R. J. Silverman of the Armour Research Foundation has been appointed to an assistant professorship; Mr. H. L. Pearson, Dr. Pasquale Porcelli, and Mr. R. E. Seall have been appointed to instructorships.

Iowa State College announces: Dr. E. H. Feller of the University of Wisconsin has been appointed to an instructorship; Dr. W. D. Lindstrom, Dr. D. E. Sanderson, Dr. R. D. Stalley, and Dr. F. M. Wright have been promoted to assistant professorships.

Kent State University reports that Assistant Professors B. B. Dressler and R. Y. Iwanchuk have been promoted to associate professorships.

Laval University announces the following: Professor Alexandre LaRue was the delegate of the University at the International Congress of Mathematicians; Dr. Arthur Dubé has been appointed to a professorship.

At Louisiana State University: Assistant Professor Heron Collins of the University of South Carolina has been appointed to an assistant professorship; Assistant Professor E. G. Kundert has been promoted to an associate professorship.

Marquette University reports: Miss Miriam Connellan, Rev. L. J. Heider, Mr. J. E. Kelley, formerly an analyst with the Department of Defense, Miss Joan Salatino, Mr. Richard Schwaller, previously a graduate assistant at the University, and Mr. Earl Swokowski, formerly a graduate assistant at the University of Wisconsin, have been appointed to instructorships.

Montana State College announces: Mr. J. L. Simpson and Dr. Hans Sagan, formerly an assistant at the University of Vienna, have been appointed to instructorships; Dr. J. E. Whitesitt has been promoted to an assistant professorship; Assistant Professor A. L. Hess has been promoted to an associate professorship; Professor Frieda M. Bull has retired with the title Professor Emeritus; Mr. D. F. DeLap has retired.

At Northwestern University: Assistant Professor M. A. Rosenlicht has received a Fulbright award for research at the University of Rome; Assistant Professor George Springer has received a Fulbright award to lecture at the University of Munster, Germany, October 1954 to June 1955; Dr. W. A. Michael, Jr., previously an assistant at the University of Illinois, and Dr. W. T. Kyner, formerly an assistant at the University of California, have been appointed to instructorships; Visiting Professor J. A. E. Dieudonné has been promoted to a professorship; Assistant Professor Daniel Zelinsky has been promoted to the position of Associate Professor; Dr. J. C. E. Dekker, Dr. M. P. Gaffney, and Dr. Alex Rosenberg have been promoted to assistant professorships.

Ohio State University reports the following: Dr. J. E. Adney, Jr., formerly an assistant at the University, Dr. J. M. Shapiro, and Mr. Clifford Spector have

been appointed to instructorships; Dr. A. D. Ziebur has been promoted to an assistant professorship.

Oklahoma Agricultural and Mechanical College announces: Dr. George Marsaglia has been appointed to an assistant professorship; Assistant Professor F. A. Graybill has been promoted to an associate professorship.

At Purdue University: Dr. L. J. Cote, formerly a graduate student at Columbia University, and Mr. S. F. Reiter, previously a research associate at Stanford University, have been appointed to assistant professorships; Dr. A. H. Copeland, Jr. of Massachusetts Institute of Technology, Mr. W. R. Fuller, formerly a mathematician with Naval Ordnance, Indianapolis, Dr. G. L. Krabbe, recently an assistant at the University of California, Dr. C. J. Neugebauer, previously an assistant at Ohio State University, Dr. C. M. Petty, formerly a research instructor at Duke University, Assistant Professor G. C. Preston of Macalester College, Mr. R. W. Randall, Jr., previously a student at Rice Institute, and Mr. D. M. Mesner, recently an assistant at Michigan State College, have been appointed to instructorships; Dr. Morris Skibinsky, research assistant at Duke University, has been appointed Research Associate in the Statistical Laboratory; Assistant Professor C. R. Putnam has been promoted to an associate professorship; Assistant Professor E. A. Trabant has been promoted to the position of Associate Professor of Mathematics and Engineering Science; Associate Professor G. H. Graves has retired with the title Professor Emeritus.

State University of Iowa reports the following: Professor M. F. Smiley, who is on leave of absence during the academic year 1954–55, spent the Fall quarter at the University of Chicago and is now at the University of Washington; Dr. H. A. Dye of the Institute for Advanced Study and Dr. H. M. Johnson of the Yerkes Observatory, University of Chicago, have been appointed to assistant professorships; Associate Professors Roscoe Woods and H. V. Price have been promoted to professorships; Professor C. C. Wylie has retired but will continue on a part-time basis.

University of Arizona announces: Assistant Professor H. D. Sprinkle of Alabama Polytechnic Institute has been appointed to an assistant professorship; Mr. D. B. Witmeyer, formerly a mathematics teacher at Duncan Arizona High School, has accepted an instructorship; Mr. D. N. Leeson, formerly a student at the University of Bridgeport, and Miss Virginia C. Clover, previously a student at Augustana College, have been appointed teaching fellows.

The University of California, Berkeley, announces: The following new appointments have been made for the year 1954–55: Visiting Associate Professor H. D. Huskey, Acting Assistant Professors S. G. Bourne and Bernard Sherman, Instructors E. A. Bishop, J. B. Butler, Jr., and H. A. Osborn; the following appointments have been made for Spring 1955: Visiting Professors A. S. Besicovitch, Charles Loewner, and N. E. Steenrod; Assistant Professor L. A. Henkin has been promoted to an associate professorship and is on leave during 1954–55

on a Fulbright fellowship; Associate Professor E. L. Lehmann has been promoted to a professorship; Acting Assistant Professor M. H. Protter has been promoted to an associate professorship; Professor Sophia L. McDonald has retired with the title of Professor Emeritus; Associate Professor R. H. Sciobereti has retired with the title of Associate Professor Emeritus; Assistant Professor L. H. Swinford has retired; Assistant Professor S. P. Diliberto is on leave of absence during 1954–55 and is at the Institute for Advanced Study; Assistant Professor Elizabeth L. Scott is on leave of absence in Europe.

University of California at Los Angeles reports: Associate Professor L. R. Sario of Massachusetts Institute of Technology has been appointed to an associate professorship; Associate Professor J. W. Green has been promoted to a professorship; Assistant Professor Alfred Horn has been promoted to an associate professorship; the following are on sabbatical leave for the year 1954–55: Professor M. R. Hestenes who is in Oslo, Norway, on a Guggenheim and Fulbright fellowship; Dr. J. D. Swift, now at the Institute for Advanced Study; Professor A. E. Taylor, who is in Geneva, Switzerland.

University of Chicago announces the following: Professor Saunders MacLane addressed the Henri Poincaré Festival in Paris, France, in October 1954; Dr. R. K. Lashof has been appointed to an instructorship; Dr. Armand Borel, formerly a member of the Institute for Advanced Study, is a visiting professor; Dr. W. H. Cockcroft, lecturer at King's College, Aberdeen, Scotland, has been appointed Visiting Lecturer.

University of Houston reports the following: Assistant Professor C. P. Benner has been appointed Chairman of the Department of Mathematics; Dr. D. H. Wright has accepted an assistant professorship.

At the University of Illinois: Professeur titulaire Roger Godement of the Universite de Nancy, France, has a position as George A. Miller Visiting Professor; Associate Professor N. S. Hawley, Jr., of Louisiana State University has been appointed to an assistant professorship; Assistant Professor M. E. Munroe has been promoted to an associate professorship.

University of Maryland announces: Dr. L. F. McAuley, part-time instructor at the University of North Carolina and Mr. W. G. Rosen, formerly a graduate assistant at the University of Illinois, have been appointed to instructorships; Assistant Professors R. A. Good, G. S. S. Ludford, and D. M. Young, Jr., have been promoted to associate professorships.

University of Massachusetts reports: Mrs. Alice Epstein and Mr. R. E. Schwartz of Ripon College have been appointed to instructorships; Mr. A. W. Wallace, formerly a student at Northeastern University, has been appointed to a part-time instructorship; Mr. R. W. Deland has been appointed a teaching fellow; Assistant Professor I. H. Rose has been promoted to an associate professorship.

The University of Oklahoma announces the following: Associate Professor Casper Goffman of Wayne University has accepted a professorship; Professor

J. O. Hassler has retired with the title of Professor Emeritus of Mathematics and Astronomy. A two day conference for mathematics teachers will be held at the University in June, 1955.

University of Rochester reports the following: Mr. W. A. Small, previously a graduate student at the University, has been appointed to an instructorship; Mr. G. C. Branche, Mr. D. M. Burton, Mr. P. L. Kingston, and Mr. T. R. Knapp have been appointed to graduate instructorships; Dr. Robert MacDowell has been promoted to an assistant professorship; Professor Wladimir Seidel is on leave and is at Notre Dame University as Visiting Professor.

At the University of Saskatchewan: Dr. Richard Blum of Acadia University and Mr. W. B. Antliff, formerly a teaching fellow at Queen's University, Ontario, have been appointed to instructorships.

University of Tennessee announces: Dr. A. A. Blank, a research associate at the University of Illinois, has been appointed to an assistant professorship; Mr. G. L. Curme, Mr. P. H. Doyle of Western Michigan College, and Mr. G. M. Speed have been appointed to instructorships.

University of Toronto reports: Professor L. J. Mordell of the University of Cambridge has been appointed Visiting Professor; Dr. Hanno Rund, seminar lecturer of Bonn University, has been appointed to an assistant professorship; Mr. R. O. A. Robinson and Mr. Ralph Wormleighton have been appointed Lecturers; Associate Professor G. deB. Robinson has been promoted to a professorship.

New York University, Washington Square College, announces the following: Assistant Professor Louis Baron has been promoted to an associate professorship; Research Assistant Professor S. C. Lowell has been promoted to a research associate professorship.

West Virginia University reports the following: Dr. E. E. Posey of the University of Tennessee has been appointed to an assistant professorship; Miss Maryelsie Hawkins, formerly a graduate assistant at the University of Illinois, has been appointed to an instructorship.

At Yale University: Dr. J. P. Jans of the University of Michigan has been appointed to an instructorship; Dr. J. T. Schwartz has been promoted to an assistant professorship.

Miss Ruth J. Abrams, formerly a technical aide for Bell Telephone Laboratories, Murray Hill, New Jersey, is a graduate student at Columbia University.

Dr. Miriam C. Ayer, member of the staff of Sandia Corporation, Albuquerque, New Mexico, has been appointed to an assistant professorship at the University of Missouri.

Associate Professor J. M. Barbour, Department of Music, Michigan State College, has been promoted to a professorship.

Mr. D. C. Barton of the University of Rochester has accepted a position as mathematician with the Eastman Kodak Company, Rochester, New York.

Dr. Gertrude Blanch, recently with the Consolidated Engineering Company,

Pasadena, California, has a position as Senior Mathematician at Wright Air Development Center, Dayton, Ohio.

Assistant Professor T. A. Botts of the University of Virginia has been promoted to an associate professorship.

Dr. Barron Brainerd has been appointed to an instructorship at the University of British Columbia.

Mr. T. F. Brophy, formerly a graduate student at New York State College for Teachers, is teaching at Stillwater Central School, New York.

Mr. L. L. Campbell of the University of Toronto has accepted a position as Scientific Service Officer with the Defense Research Board, Ottawa, Ontario, Canada.

Assistant Professor Harvey Cohn of Wayne University has been promoted to an associate professorship.

Dr. Margaret F. Conroy of Purdue University has accepted a position as Assistant Professor at Washington University.

Professor H. H. Conwell, Emeritus Dean of Beloit College, is Visiting Professor at Rockford College.

Mr. J. R. Cox, formerly a student at Lebanon Valley College, is now a graduate assistant at Pennsylvania State University.

Mr. H. B. Curtis, Jr. of Texas Agricultural and Mechanical College has been promoted to an assistant professorship.

Mr. J. M. Elkin, previously chief statistician for the Railroad Retirement Board, Chicago, Illinois, is employed as Chief Statistician by Martin E. Segal & Company, New York City.

Mr. J. G. Elliott of Ohio University is employed by Bendix Aircraft Company, Detroit, Michigan.

Assistant Professor D. H. Erkiletian, Jr. of Missouri School of Mines has been promoted to an associate professorship.

Professor G. M. Ewing of the University of Missouri is on leave of absence for the academic year 1954–55 and is a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Professor C. B. Gass, head of the Mathematics Department of Nebraska Wesleyan University, has been appointed to an associate professorship at DePauw University.

Mr. I. B. Goldberg, formerly a student at New York University, is a graduate student at Harvard University.

Dr. Harry Gonshor of the University of Southern California has accepted an assistant professorship at the University of Miami.

Mr. Basil Gordon, formerly a junior instructor at Johns Hopkins University, has been appointed a graduate assistant at the California Institute of Technology.

Mr. A. S. Gregory, formerly a student at the University of Illinois, is now a fellow at the University of Chicago.

Mr. Arshag Hajian, recently a student at the University of Chicago, is now a graduate student at Yale University.

Dr. J. K. Hale of Purdue University has accepted a position as a mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Mr. G. R. Ingram, formerly of Montana State College, is employed by the Bureau of Public Roads, Helena, Montana.

Mr. A. A. Karwath of St. Ambrose College has been appointed to an instructorship at the University of Iowa.

Mr. R. R. Kemp, recently a student at McMaster University, is now a teaching assistant at Massachusetts Institute of Technology.

Mr. W. H. Leser has been appointed to an instructorship at Franklin and Marshall College.

Dr. R. W. Long, formerly a lecturer at the University of Pittsburgh, is employed now as a mathematician by Westinghouse Electric Corporation, East Pittsburgh, Pennsylvania.

Mr. W. M. Lowney, previously a graduate student at Montana State College, is now Acting Instructor at the State College of Washington.

Mr. J. F. Manogue, formerly a student at the Catholic University of America, is a graduate student at Columbia University.

Assistant Professor Elna B. McBride of Memphis State College has been promoted to an associate professorship.

Dr. R. W. McKelvey of Purdue University is now a post-doctoral fellow at the University of Maryland.

Mr. C. E. Moulton, formerly a graduate student at the University of Buffalo, has been appointed to a professorship at Shurtleff College.

Professor F. S. Nowlan of the University of Illinois has been appointed Visiting Professor at the College of William and Mary.

Assistant Professor Anne F. O'Neill of Wheaton College, Norton, Massachusetts, has been promoted to the position of Associate Professor and Acting Head of the Department of Mathematics during 1954–55.

Assistant Professor D. B. Owen of Purdue University has a position as Statistician at the Sandia Corporation, Albuquerque, New Mexico.

Visiting Associate Professor T. K. Pan of the University of Oklahoma has been appointed to an associate professorship.

Mr. E. C. Paxhia, formerly a student at the University of Rochester, is now a graduate assistant at Washington University.

Mr. E. I. Pina, recently a part-time instructor at the University of Massachusetts, is now a junior engineer with Boeing Airplane Company, Seattle, Washington.

Professor Emeritus W. W. Rankin of Duke University has been appointed to an instructorship at the Phillips Exeter Academy, New Hampshire, for the academic year 1954–55.

Mr. P. D. Ritger, previously a research assistant in mathematics at New

York University, has been appointed to an instructorship at Stevens Institute of Technology.

Dr. Samuel Schecter of Lehigh University has a position as research associate at New York University, Institute of Mathematical Sciences.

Professor J. P. Scholz of Western College has been appointed Head of the Department of Mathematics.

Dr. Esther Seiden, formerly with the Committee on Statistics, University of Chicago, is now Assistant Professor at Howard University.

Mr. R. J. Semple, formerly a part-time instructor at Princeton University, has been appointed Lecturer at Carleton College, Ottawa, Canada.

Sister M. Walter Reginald, an instructor at the College of Saint Teresa, has been appointed to an instructorship at Dominican College, Wisconsin.

Dr. N. B. Smith, formerly a graduate student at Iowa State College, has been appointed to an instructorship at San Diego State College.

Mr. R. C. Stewart of Trinity College, Connecticut, has been promoted to an assistant professorship.

Mr. C. R. Strain, previously a mathematician for Engineering Research Associates, Arlington, Virginia, is now a programmer and digital computer for Remington-Rand, New York City.

Mr. C. H. Taft of West Virginia University has accepted a position as mathematician with the National Security Agency, Washington, D. C.

Assistant Professor D. L. Thomsen, Jr. of the Pennsylvania State University has accepted a position as Applied Science Representative with International Business Machines Corporation, Philadelphia, Pennsylvania.

Mr. D. E. Thoro, previously a mathematician for R. C. A. Service Company, Cocoa, Florida, has been appointed to an instructorship at the University of Florida.

Assistant Professor G. L. Walker of Purdue University is now Head of the Mathematical Section of the American Optical Company, Southbridge, Massachusetts.

Mr. W. G. Weideman, formerly in military service, is now a graduate student at the University of Michigan.

Miss Irene S. Welna, recently a student at Saint Joseph College, is teaching at Bloomfield Junior High School, Connecticut.

Assistant Professor C. S. Wolfe of Shepherd College is now a teaching assistant at the University of Maryland.

Mr. P. W. Zehna of Colorado State College of Education has been appointed to an instructorship at the University of Kansas.

Dr. A. Zirakzadeh of the University of Colorado has been appointed to an instructorship at the State College of Washington.

Assistant Professor J. V. Limpert of St. Lawrence University died on September 2, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 61 persons have been elected to membership by the Board of Governors on applications duly certified.

A/2C Ernest Anderson, Keesler Air Force Base, Mississippi.

P. G. Archer, B. A. (Buffalo) Mathematician, Bell Aircraft Corporation, Niagara Falls, N. Y.

R. H. Ayers, Student, Lebanon Valley College, Annville, Pa.

G. E. Bardwell, M.S. (Colorado) Research Asso., University of Denver.

Joan M. Beauchamp, B.A. (Keuka) Teacher, Middlesex Valley Central School, Rushville, N. Y.

A. H. Blessing, B.A. (Buffalo) Instr., University of Buffalo.

W. J. Blundon, M.A. (Columbia U.) Professor, Memorial University of Newfoundland, St. John's, Newfoundland, Canada.

L. L. Brassaw, Jr., B.S. (U. S. Naval Academy) Teaching Fellow, University of Buffalo.

Rev. A. E. Cahill, O.S.B., A.B. (St. Benedict's) Instr. in Physics and Mathematics, Belmont Abbey College, Belmont, N. C.

Lamberto Cesari, Ph.D. (Pisa) Professor, Purdue University.

C. Y. Chao, M.S. (State U. of Iowa) Instr., Coe College.

Ella L. Clement, A.B. (State U. of Iowa) Teacher, Douglas High School, Oklahoma City, Okla.

Miriam E. Connellan, M.A. (Catholic U.) Instr., Marquette University.

D. C. Davis, B.A. (Oklahoma) Grad. Asst., University of Oklahoma.

P. A. Davis, Student, University of British Columbia.

Gertrude V. Decker, M.A. (Hofstra) Teacher, East Rockaway High School, N. Y.

E. T. Denmark, Jr., M.S. (Florida S. U.) Instr., Chipola Junior College, Marianna, Fla.

Mary G. DiMaggio, A.B. (Montclair S. T. C.) Teacher, Closter High School, N. J.

J. E. Dixon, Student, William Jewell College.

E. E. Dougherty, Student, Sacramento State College.

Willard Draisin, Student, Brooklyn College.

Mrs. Joy B. Easton, M.S. (W. Va. U.) Secretary, Mathematics Department, West Virginia U.

E. E. Floyd, Ph.D. (Virginia) Asso. Professor, University of Virginia.

C. L. Gape, B.A. (Buffalo) Instr., University of Buffalo.

Leonard Gillman, Ph.D. (Columbia U.) Asst. Professor, Purdue University.

William Granet, M.A. (Columbia U.) Asst. to Director, Office of Statistical & Research Services, Boston University.

S. L. Greitzer, M.A. (Columbia U.) Teacher, Yeshiva University and Bronx High School of Science.

L. W. Gunter, M.S. (Wisconsin) Instr., Western Michigan College of Education.

A. B. Harper, Jr., S.B. (M.I.T.) Analyst, Arthur D. Little Company, Nashua, N. H.

Robert Hayes, M.A. (Columbia U.) Chairman, Mathematics Department, Hempstead High School, N. Y.

R. L. Helmbold, M.S. (Carnegie I. T.) Teaching Asst., Carnegie Institute of Technology.

Lesta Hoel, M.A. (Oregon) Supervisor of Mathematics, Portland, Oregon.

A. A. J. Hoffman, Student, University of Texas.

Harriett R. Junior, M.S. (Howard) Instr., Hampton Institute.

J. G. Kemeny, Ph.D. (Princeton) Professor, Dartmouth College.

Joachim Lambek, Ph.D. (McGill) Asso. Professor, McGill University.

W. H. Leser, A.M. (Pennsylvania) Instr., Franklin & Marshall College.

EDNA M. LOVE, M.A. (Pittsburgh) Teacher, Derry Township High School, Derry, Pa.

S. F. MACK, Ph.D. (California) Asst. Professor, Pennsylvania State University.

R. C. MEACHAM, Ph.D. (Brown) Asso. Professor, University of Florida.

R. A. MOORE, Ph.D. (Washington U.) Instr., Yale University.

J. P. NOLAN, Norman, Oklahoma.

R. J. OEDY, A.B. (Indiana) Grad. Student, Iowa State College.

J. H. ORRICK, B.S.M.E. (Lawrence I. T.) Engr., Chrysler Corp.; Instr., Lawrence Institute of Technology, Detroit, Michigan.

PROM PANITCHPAKDI, M.S. (Chicago) Grad. Student, University of Kansas.

MRS. MARIE W. PEACOCK, B.A. (U. of Washington) Teacher, Lincoln High School, Seattle, Wash.

J. M. PELLEGRINO, Student, Siena College, Loudonville, N. Y.

R. O. A. ROBINSON, M.A. (Cambridge) Lecturer in Applied Math., University of Toronto.

S. G. SADLER, D.Ed. (Florida) Asso. Professor, University of Florida.

R. W. SCHENKEL, B.S. (Lawrence I. T.) Instr., Lawrence Institute of Technology, Detroit, Mich.

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NANCY M. SCRIBANO, M.S. (Northwestern) Instr., Ohio University.

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J. L. SIMPSON, M.S. (Montana S. C.) Instr., Montana State College.

SISTER MARY VERA, B.V.M., M.A. (Catholic U.) Chairman, Mathematics Department, Mundelein College, Chicago, Illinois.

GRANT SMITH, A.B. (Columbia U.) Asso. Professor, Philadelphia Textile Institute.

J. R. VANSTONE, Student, University of Toronto.

D. L. VOGELSANG, A.B. (W. Va. U.) Asst. Inst. and Grad. Student, West Virginia University.

W. J. WALBESSER, M.S. (Stevens I. T.) Electronics Engr., Cornell Aeronautical Lab., Buffalo, N. Y.

GWENDOLYN B. WENMAN, A.B. (Montclair S. T. C.) Teacher, Madison High School, N. J.

K. B. WILLIAMS, Student, Sacramento State College.

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The October meeting of the Minnesota Section of the Mathematical Association of America was held at the University of Manitoba in Winnipeg, Manitoba, on October 16, 1954. Sessions were held in the forenoon, at luncheon and in the afternoon. Professors J. W. Lawson, R. P. Winter and Sister M. Leontius, Chairman of the Section, presided at the respective sessions.

Forty-seven persons attended the meeting including the following thirty members of the Association:

F. J. Arena, J. M. Calloway, C. S. Carlson, N. J. Divinsky, R. J. Dowling, K. L. Hankerson, Hildegarde H. Howden, Diane M. Johnson, G. K. Kalisch, J. W. Lawson, W. S. Loud, Walter Lyche, K. O. May, W. H. McBride, W. H. McEwen, W. R. McEwen, N. S. Mendelsohn, E. O. Nelson, J. C. Peterson, P. A. Rognlie, P. C. Rosenbloom, L. W. Sheridan, Sister M. Thomas à Kempis, Sister Mary Leontius, F. C. Smith, R. C. Staley, O. E. Stanaitis, K. W. Wegner, R. P. Winter, F. L. Wolf.

By invitation of the Executive Committee, Professor G. K. Kalisch of the University of Minnesota delivered an address at the morning session entitled "Canonical Forms of Finite and Infinite Matrices." The abstract of this address follows:

This address is devoted principally to a discussion of certain canonical forms of certain finite and infinite matrices arising from linear transformations of bilinear or quadratic forms in finite or infinite complex or real Hilbert spaces. The following five equivalence relations and certain associated canonical forms—sometimes confined to special classes of matrices—were discussed: (1) equivalence ($M\sim M'=PMQ$, P and Q non-singular); (2) unitary (orthogonal) equivalence ($M\sim M'=UMV$, U and V unitary (orthogonal)); (3) similarity ($M\sim M'=P^{-1}MP$, P non-singular); (4) conjunctivity (congruence) ($M\sim M'=P^*MP$, P non-singular with complex (real) entries); (5) unitary (orthogonal) similarity ($M\sim M'=U^*MU$, U unitary (orthogonal)). A number of unsolved problems connected with the preceding equivalence relations were also briefly discussed. In the infinite case, some attention was paid to certain sequence spaces other than l_2.

The following short papers were presented:

1. *A lemma on positive definite matrices*, by Professor E. O. Nelson, University of North Dakota.

The purpose of this paper is to show that a matrix

$$B=\begin{pmatrix} a_{11}\cdots a_{1k} & b_1 \\ \cdots\cdots\cdots & \cdots \\ a_{k1}\cdots a_{kk} & b_k \\ b_1 \cdots b_k & b_{k+1} \end{pmatrix}$$

formed by bordering a positive definite k by k matrix

$$A=\begin{pmatrix} a_{11}\cdots a_{1k} \\ \cdots\cdots \\ a_{k1}\cdots a_{kk} \end{pmatrix}$$

with any constants $b_1, \cdots, b_{k+1}$ is also positive definite if the determinant of the matrix B is greater than zero.

2. *The use of statistics in plastic film evaluation*, by Dr. L. W. Sheridan, General Mills, Minneapolis, Minnesota.

This paper presents some statistical methods considered in arriving at numerical values from laboratory tests for use in writing some of the specifications for polyethylene film used in fabricating stratospheric balloons.

3. *A note on an integral test for convergence of series of arbitrary terms*, by Professor O. E. Stanaitis, St. Olaf College.

Let $f(x)$ denote a function of a real variable on the interval $1\leqq x<\infty$. If $f'(x)$ exists and is integrable over the same interval, and if $\int_1^\infty f'(x)dx$ is absolutely convergent, then the series $\sum_{x=1}^\infty f(x)$ and the integral $\int_1^\infty f(x)dx$ converge or diverge together. It has been shown that the theorem holds if a derivative $f^{(n)}(x)(n>1)$ exists and the integral $\int_1^\infty f^{(n)}(x)dx$ is absolutely convergent.

4. *Nilpotent matrices*, by Professor B. Noonan, University of Manitoba, introduced by Professor J. W. Lawson.

The main theorem demonstrates that properly triangular square matrices with elements in a field and sets of such matrices are nilpotent. The proofs are made elementary by considering only the principal diagonals of the matrices, namely, the first secondary diagonal in each matrix containing a non-zero element. It is further shown that in any set of properly triangular n by n matrices of nilpotency $n-r$, all matrices having the rth secondary diagonal as principal diagonal have at

least one zero element in this principal diagonal, and it is the same element in all such matrices. The converse theorem is also given.

5. *On a construction of the regular pentagon*, by Professor F. J. Arena, North Dakota Agricultural College.

In this paper the author gives a construction of the regular pentagon which is not well-known. The construction is as follows:

In any circle with center at O, let AOB and COD be two perpendicular diameters. Bisect OB at E. With EC as radius and E as center, strike an arc intersecting OA at F. Then in the circle $CADB$ draw the chords CG, GH, HI and IJ all equal to CF and $CGHIJC$ is the required pentagon.

The author then proves that the polygon $CGHIJC$ is a regular pentagon.

6. *Some simple but significant calculus problems in economics*, by Professor K. O. May, Carleton College.

If y is the total costs of a firm and x its total output, it is quite realistic to assume that $y=mx+b$ where b is the overhead and m is the marginal cost. If the best use of existing resources is made, m and b can be changed only by modifying technology, methods or organization. Typically, $m=f(b)$, where $f'(b)<0$ and $f''(b)>0$. It is easy to determine the best cost function for a given output, *i.e.*, that which minimizes cost for that output. The real problem, however, is to choose a cost function best suited to some anticipated time series of outputs in the future. It is a simple calculus problem to show that the cost over a period of time is minimized by choosing the cost function that yields minimum cost for the anticipated average output. This yields also the maximum profit. The problem may be modified by assuming that cost is some non-linear function. By defining total costs and average outputs in terms of integrals, one may acquaint the student with economic interpretations of mathematical manipulations usually associated with centroids and moments. Indeed for a quadratic cost function, it is easily shown that the best cost function is determined by the mean and the variance of the outputs.

7. *Matrices poorly behaved with respect to the computation of their inverses*, by Professor N. S. Mendelsohn, University of Manitoba.

If A is a square matrix of real numbers with non-vanishing determinant, it has a unique left inverse which is also a right inverse. For some matrices A there are matrices X^* which are almost left inverses but which are very poor as right inverses. In fact we can establish the following theorem:

Let ϵ and K be arbitrary positive numbers. There exist, for each $n\geqq 2$, n by n matrices X and A such that every element of XA differs from the corresponding element of the identity matrix by less than ϵ while every element of AX is larger than K.

A connection between the numbers ϵ, K and the separation of the characteristic values of A is obtained.

8. *Sequences illustrating uniform and dominated convergence*, by Professor W. S. Loud, University of Minnesota.

If a sequence of functions $\{f_n(x)\}$ has a pointwise limit $f(x)$ on $a\leqq x\leqq b$, sufficient conditions for $\int_a^b f_n(x)dx$ to approach $\int_a^b f(x)dx$ (assuming all integrals exist) are that the convergence be uniform, or that the convergence be uniform on every closed subinterval with the sequence being dominated by an integrable function on the entire interval. Taking α, β, γ and δ as positive, the sequence $f_n(x)=n^\alpha x^\beta e^{-n^\gamma x^\delta}$ on $0\leqq x\leqq 1$ converges pointwise to zero. It is uniformly convergent if $\alpha\delta<\beta\gamma$, and satisfies the second condition if $\alpha\delta<(\beta+1)\gamma$. It can be verified that the integrals converge to zero if $\alpha\delta<(\beta+1)\gamma$. A sequence having similar properties on the same interval is $f_n(x)=n^\alpha x^\beta/(1+n^\gamma x^\delta)$ where it is assumed in addition that $\alpha<\gamma$.

9. *A new definition of the radical*, by Professor N. J. Divinsky, University of Manitoba.

This note is a report on the recent Russian definition of the Brown and McCoy radical. It considers papers by Andrunakievich in 1948 and 1952 and by Kurochkin in 1949. Use is made of co-united multiplication: $a \circ b = a+b-ab$; and the strange concept of a co-united right ideal I which is a subset of a ring A such that $a \circ x$ is in I for every a in I and x in A and such that for every a, b and c in I, $a-b+c$ is in I. The set I is not a right ideal in the ordinary sense; in fact, it is not closed with respect to addition. All the usual theorems on the radical are proved using this new point of view.

F. C. SMITH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 7, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May 7, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS, Fort Hays Kansas State College, Hays, March 26, 1955.

KENTUCKY, Georgetown College, Georgetown, April 30, 1955.

LOUSIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 18–19, 1955.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, Maryland, April 16, 1955.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, March 26, 1955.

MINNESOTA, College of St. Teresa, Winona, Minnesota, May 7, 1955.

MISSOURI, University of Kansas City, April 22, 1955.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1955.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 23, 1955.

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, April 22–23, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April 22–23, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May 7, 1955.

EMPLOYMENT OPPORTUNITIES

The University of Alberta, Edmonton, Alberta. (1) Assistant or Associate Professor of statistics; appointment to be effective September 1, 1955. (2) Sessional Lecturer in Mathematics for eight months from September 1, 1955; prospect of permanent appointment. Applications including transcript of academic record, curriculum *vitae*, publications, names of two references, and recent photograph or snapshot to be sent to Walter H. Johns, Dean, Faculty of Arts and Science, University of Alberta.

University of British Columbia, Vancouver 8, Canada. Instructor, Ph.D., Mathematics Department.

By action of the Board of Governors this Employment Opportunities section will be discontinued. Employers are referred to the Employment Registrar now maintained at all principal meetings of the mathematical organizations.

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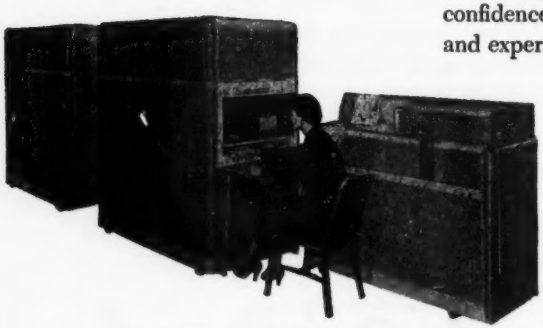

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